

# Non-negative integral level affine Lie algebra tensor categories and their associativity isomorphisms

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## Abstract

For a finite-dimensional simple Lie algebra  $\mathfrak{g}$ , we use the vertex tensor category theory of Huang and Lepowsky to identify the category of standard modules for the affine Lie algebra  $\widehat{\mathfrak{g}}$  at a fixed level  $\ell \in \mathbb{N}$  with a certain tensor category of finite-dimensional  $\mathfrak{g}$ -modules. More precisely, the category of level  $\ell$  standard  $\widehat{\mathfrak{g}}$ -modules is the module category for the simple vertex operator algebra  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ , and as is well known, this category is equivalent as an abelian category to  $\mathbf{D}(\mathfrak{g}, \ell)$ , the category of finite-dimensional modules for the Zhu's algebra  $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ , which is a quotient of  $U(\mathfrak{g})$ . Our main result is a direct construction using Knizhnik-Zamolodchikov equations of the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$  induced from the associativity isomorphisms constructed by Huang and Lepowsky in  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$ . This construction shows that  $\mathbf{D}(\mathfrak{g}, \ell)$  is closely related to the Drinfeld category of  $U(\mathfrak{g})[[\hbar]]$ -modules used by Kazhdan and Lusztig to identify categories of  $\widehat{\mathfrak{g}}$ -modules at irrational and most negative rational levels with categories of quantum group modules.

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# 1 Introduction

Suppose  $\mathfrak{g}$  is a finite-dimensional simple Lie algebra over  $\mathbb{C}$ ; then the affine Lie algebra  $\widehat{\mathfrak{g}}$  is a central extension of the loop algebra  $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$  by a one-dimensional space  $\mathbb{C}\mathbf{k}$ . If  $\mathbf{k}$  acts on a  $\widehat{\mathfrak{g}}$ -module by a scalar  $\ell \in \mathbb{C}$ , we say that the module has level  $\ell$ . Categories of  $\widehat{\mathfrak{g}}$ -modules at fixed non-negative integral levels are particularly important in physics, since they correspond to WZNW models, important examples of rational conformal field theories. The study of conformal field theory by physicists, especially by Moore and Seiberg, predicted that the category of standard (that is, integrable highest weight)  $\widehat{\mathfrak{g}}$ -modules at a fixed level  $\ell \in \mathbb{N}$  should have the structure of a rigid braided tensor category (see for instance [3], [43], and [50]). Indeed, in [38] and [39], Kazhdan and Lusztig showed that when  $\ell \notin \mathbb{Q}$ , or when  $\ell \in \mathbb{Q}$  and  $\ell < -h^\vee$  where  $h^\vee$  is the dual Coxeter number of  $\mathfrak{g}$ , a certain category of  $\widehat{\mathfrak{g}}$ -modules of level  $\ell$  has a natural braided tensor category structure, and they proved rigidity for most of these tensor categories in [41]. However, their constructions do not apply to the case  $\ell \in \mathbb{N}$ .

There are several approaches to obtaining tensor category structure when  $\ell \in \mathbb{N}$ . First, motivated by Kazhdan and Lusztig's constructions, Huang and Lepowsky developed a general tensor product theory for the category of modules for a vertex operator algebra in [22]-[24] and [17]. Since the category of standard  $\widehat{\mathfrak{g}}$ -modules at a fixed level  $\ell \in \mathbb{N}$  is the module category for a simple vertex operator algebra  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$  ([16]), they were able to use this theory in [25] to prove that this category has natural braided tensor category structure; rigidity, and indeed modularity, of this braided tensor category was proved in [21]. More recently, in [26]-[33], Huang, Lepowsky, and Zhang have developed a more general logarithmic tensor category theory for so-called generalized modules for a vertex operator algebra. Using this theory, Zhang showed in [53] that the braided tensor categories of Kazhdan and Lusztig when  $\ell \notin \mathbb{Q}$  or  $\ell \in \mathbb{Q}_{< -h^\vee}$  agree with the vertex algebraic braided tensor categories of certain generalized  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules.

Another approach to the  $\ell \in \mathbb{N}$  case using ideas of Beilinson, Feigin, and Mazur yields braided tensor category structure (see for instance Chapter 7 of [2]), but not rigidity. A third approach by Finkelberg in [12], [13] involves transferring Kazhdan and Lusztig's constructions at negative level to the positive level categories. This work also requires the Verlinde formula for multiplicities of irreducible modules in tensor products, which was proved inde-

pends by Faltings [10] and Teleman [52] for  $\widehat{\mathfrak{g}}$ -modules and was proved by Huang [20] in a general vertex algebraic context. Finkelberg's work does not apply to the cases  $E_6$  level 1,  $E_7$  level 1, and  $E_8$  levels 1 and 2 because Kazhdan and Lusztig did not prove rigidity for the corresponding negative level categories.

All the constructions of tensor category structure on  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$ , that is, the category of standard  $\widehat{\mathfrak{g}}$  modules at level  $\ell \in \mathbb{N}$ , are complicated by the fact that the usual vector space tensor product of two modules does not have a natural module structure. Note, for instance, that the usual Lie algebra tensor product of  $\widehat{\mathfrak{g}}$ -modules does not preserve level. This in turn means the associativity isomorphisms in this tensor category are highly non-trivial (see for instance [17] and [31]). As a result, useful, explicit descriptions of the tensor category structure on  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  are missing from the literature. In this paper, we use the vertex tensor category theory of Huang and Lepowsky to give an explicit description of the tensor category  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  when  $\ell \in \mathbb{N}$ , in particular a description of the associativity isomorphisms. More precisely, we show that there is an equivalence between  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  and an explicit tensor category  $\mathbf{D}(\mathfrak{g}, \ell)$  of finite-dimensional  $\mathfrak{g}$ -modules. We expect that this description will be useful for obtaining a uniform proof of the braided tensor equivalence between  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  and a category of quantum group modules, which we recall now.

In addition to constructing braided tensor categories of  $\widehat{\mathfrak{g}}$ -modules, Kazhdan and Lusztig showed in [40] and [41] that their category at level  $\ell$  is equivalent to the braided tensor category of finite-dimensional modules for the quantum group  $U_q(\mathfrak{g})$ , where  $q = e^{\pi i/m(\ell+h^\vee)}$ ; here  $m$  is the ratio of the squared length of the long roots of  $\mathfrak{g}$  to the squared length of the short roots. Then Finkelberg's work in [12], [13] showed that, with the possible exceptions of  $E_6$  level 1,  $E_7$  level 1, and  $E_8$  levels 1 and 2, the category of standard  $\widehat{\mathfrak{g}}$ -modules of level  $\ell \in \mathbb{N}$  is equivalent to a certain semisimple subquotient of the corresponding category of finite-dimensional quantum group modules. As mentioned above, the exceptions exist because of the use of Kazhdan and Lusztig's constructions at negative levels. Thus to prove the equivalence between  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  when  $\ell \in \mathbb{N}$  and the quantum group category with no exceptions, we need an explicit description of the (rigid) tensor category structure at non-negative level.

We now discuss the results of this paper in more detail. Since  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  when  $\ell \in \mathbb{N}$  is semisimple, it follows from [54] that there is an equivalence of abelian categories between  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  and the category of finite dimensional modules for the Zhu's algebra  $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$  which takes an irreducible  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module to its lowest conformal weight space. Since  $A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_\theta^{\ell+1} \rangle$  where  $x_\theta$  is a root vector corresponding the longest root  $\theta$  of  $\mathfrak{g}$  by [16],  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  is equivalent as an abelian category to the category  $\mathbf{D}(\mathfrak{g}, \ell)$  whose objects are finite-dimensional  $\mathfrak{g}$ -modules on which  $x_\theta^{\ell+1}$  acts trivially. Then we can use this equivalence to transfer the tensor category structure on  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  constructed by Huang and Lepowsky to  $\mathbf{D}(\mathfrak{g}, \ell)$ , and it remains to give an explicit description of the tensor products, unit isomorphisms, and associativity isomorphisms thus induced in  $\mathbf{D}(\mathfrak{g}, \ell)$ .

The tensor products and unit isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$  are straightforward to describe. Since tensor products of vertex operator algebra modules are defined using a universal property involving intertwining operators (in analogy with the definition of a tensor product of vector spaces in terms of a universal property involving bilinear maps; see for instance the introduction to [26]), the description of the space of intertwining operators among irreducible  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules given in [16] allows us to identify the tensor product of two modules  $U_1$  and

$U_2$  in  $\mathbf{D}(\mathfrak{g}, \ell)$  as a certain quotient  $U_1 \boxtimes U_2$  of the usual tensor product  $\mathfrak{g}$ -module. Such a quotient is necessary because  $\mathbf{D}(\mathfrak{g}, \ell)$  is not closed under the usual tensor product of  $\mathfrak{g}$ -modules. We remark that we could obtain the tensor product  $U_1 \boxtimes U_2$  in  $\mathbf{D}(\mathfrak{g}, \ell)$  simply by taking a direct sum of irreducible  $\mathfrak{g}$ -modules with multiplicities calculated using the Verlinde formula ([10], [52], [20]) or using a result such as Theorem 6.2 in [11]. However, this would be less natural than our approach because it would leave the relation of  $U_1 \boxtimes U_2$  to  $U_1 \otimes U_2$  unclear, and it would make it more difficult to understand the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$ . The unit object of  $\mathbf{D}(\mathfrak{g}, \ell)$  is the trivial one-dimensional  $\mathfrak{g}$ -module  $\mathbb{C}\mathbf{1}$ , and the unit isomorphisms are obvious.

Most of the work in this paper focuses on describing the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$ . They cannot be trivial because if  $U_1, U_2$ , and  $U_3$  are modules in  $\mathbf{D}(\mathfrak{g}, \ell)$ ,  $U_1 \boxtimes (U_2 \boxtimes U_3)$  and  $(U_1 \boxtimes U_2) \boxtimes U_3$  are typically different (albeit isomorphic) quotients of  $U_1 \otimes U_2 \otimes U_3$ . Our main result is that the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$  come from solutions of Knizhnik-Zamolodchikov (KZ) equations ([43]), as in a category of modules for the trivial deformation  $U(\mathfrak{g})[[h]]$  of the universal enveloping algebra of  $\mathfrak{g}$ , where  $h$  is a formal variable, constructed by Drinfeld ([7], [8], [9]; see also [2], [37]). (Drinfeld's category is equivalent to a category of modules for the formal quantum group  $U_h(\mathfrak{g})$ , and Kazhdan and Lusztig used an explicit equivalence between these categories in [40], [41] to show the equivalence between their category of  $\widehat{\mathfrak{g}}$ -modules and the category of  $U_q(\mathfrak{g})$ -modules.)

To describe the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$ , consider objects  $U_1, U_2$ , and  $U_3$  in  $\mathbf{D}(\mathfrak{g}, \ell)$  and the one-variable KZ equation

$$(\ell + h^\vee) \frac{d}{dz} \varphi(z) = \left( \frac{\Omega_{1,2}}{z} - \frac{\Omega_{2,3}}{1-z} \right) \varphi(z) \quad (1.1)$$

where  $\varphi(z)$  is a  $(U_1 \otimes U_2 \otimes U_3)^*$ -valued analytic function, and  $\Omega_{1,2}, \Omega_{2,3}$  are certain (non-commuting) operators on  $(U_1 \otimes U_2 \otimes U_3)^*$ . In the case  $\ell \notin \mathbb{Q}$ , any solution  $\varphi(z)$  of (1.1) can be expressed as

$$\varphi(z) = z^{\Omega_{1,2}/(\ell+h^\vee)} \cdot \varphi_0(z)$$

around  $z = 0$ , where  $\varphi_0(z)$  is analytic in a neighborhood of 0, and as

$$\varphi(z) = (1-z)^{\Omega_{2,3}/(\ell+h^\vee)} \cdot \varphi_1(z)$$

around  $z = 1$ , where  $\varphi_1(z)$  is analytic in a neighborhood of 1. The solution  $\varphi(z)$  is completely determined by the initial value  $\varphi_0(0)$ , and also by the initial value  $\varphi_1(1)$ . Then there is a unique automorphism  $\Phi_{KZ}$  of  $(U_1 \otimes U_2 \otimes U_3)^*$ , called the Drinfeld associator, that maps the initial value  $\varphi_0(0)$  to  $\varphi_1(1)$  for any solution  $\varphi(z)$  of (1.1). In the case  $\ell \in \mathbb{Q}$ , the situation is not so simple because expansions of solutions to (1.1) about the singularities 0 and 1 may contain logarithms. However, we show that series solutions around the singularities remain determined by initial data from  $(U_1 \otimes U_2 \otimes U_3)^*$ , and we construct a Drinfeld associator  $\Phi_{KZ}$  that maps the initial datum for any solution at the singularity 0 to the initial datum at 1.

Our main theorem is that when  $\ell \in \mathbb{N}$ , the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$  are induced by adjoints of Drinfeld associators. In particular, the adjoint  $\Phi_{KZ}^*$  of  $\Phi_{KZ}$  induces a well-defined isomorphism between the two quotients  $U_1 \boxtimes (U_2 \boxtimes U_3)$  and  $(U_1 \boxtimes U_2) \boxtimes U_3$  of  $U_1 \otimes U_2 \otimes U_3$ . This assertion is not obvious from the construction of the Drinfeld associator.

Rather, it follows from the existence of associativity isomorphisms in  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  proven in [25], which is equivalent to the convergence and associativity of intertwining operators among  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules (see [17] or [31]). The associativity of intertwining operators follows from the fact (first shown in [43]) that a product of intertwining operators

$$u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \mapsto \langle u'_{(4)}, \mathcal{Y}_1(u_{(1)}, 1) \mathcal{Y}_2(u_{(2)}, 1 - z) u_{(3)} \rangle, \quad (1.2)$$

when  $u_{(1)}$ ,  $u_{(2)}$ ,  $u_{(3)}$ , and  $u'_{(4)}$  are lowest-conformal-weight vectors of their respective modules, defines a solution of (1.1) expanded about the singularity  $z = 1$ , while an iterate of intertwining operators

$$u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \mapsto \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, z) u_{(2)}, 1 - z) u_{(3)} \rangle \quad (1.3)$$

corresponds to a solution of (1.1) expanded about the singularity  $z = 0$ . From this, the definitions imply that the Drinfeld associator maps the initial datum of an iterate functional to the initial datum of a corresponding product functional. Our main theorem then follows from the (non-trivial) fact that the initial data determining the series expansions (1.2) and (1.3) are given by replacing all intertwining operators with their projections to the lowest conformal weight spaces of their target modules.

Our description of the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$  might be expected since a similar result holds for the Kazhdan-Lusztig category of  $\widehat{\mathfrak{g}}$ -modules when  $\ell \notin \mathbb{Q}$  (see for instance the discussion in Section 1.4 of [2]). However, there are several complications that make the case  $\ell \in \mathbb{N}$  more difficult. First, it is somewhat more complicated to construct Drinfeld associators when  $\ell \in \mathbb{Q}$  because expansions of solutions to the KZ equation (1.1) about its singularities typically contain logarithms. Because of this, it is more difficult to identify the initial data at the singularities determining a KZ solution corresponding to a product or iterate of intertwining operators. Identifying these initial data requires a theorem restricting the weights of irreducible standard  $\widehat{\mathfrak{g}}$ -modules. Moreover, we use the vertex tensor category structure on  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  from [25] to show that adjoints of Drinfeld associators are isomorphisms between the correct quotients of triple tensor products of  $\mathfrak{g}$ -modules in  $\mathbf{D}(\mathfrak{g}, \ell)$ . It seems to be difficult to prove directly that  $\mathbf{D}(\mathfrak{g}, \ell)$ , with its correct tensor product, is a tensor category, without using the tensor category structure on  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$ .

Although we focus in this paper on tensor categories of non-negative integral level affine Lie algebra modules, it is interesting to consider whether the methods and results here extend to  $\ell \notin \mathbb{N}$ . First, we remark that our methods and results easily extend to the case of generic level,  $\ell \notin \mathbb{Q}$ . Here, we consider the semisimple category generated by irreducible  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules, called  $\mathcal{O}_{\ell+h^\vee}$  in [38]-[41]. It is easy to use results in [16] and [47] and the methods of this paper to show that the vertex algebraic tensor category structure on  $\mathcal{O}_{\ell+h^\vee}$  constructed in [53] based on [26]-[33] is equivalent to a tensor category whose objects consist of all finite dimensional  $\mathfrak{g}$ -modules, whose tensor product is the usual tensor product of  $\mathfrak{g}$ -modules, and whose associativity isomorphisms are obtained from Drinfeld associators constructed from solutions of KZ equations. These results may be compared with the tensor category structure on  $\mathcal{O}_{\ell+h^\vee}$  constructed in [38]-[41] (see also Section 1.4 of [2]).

For the case  $\ell \in \mathbb{Q}_{<-h^\vee}$ , our methods and results do not immediately generalize to yield a description of the tensor category structure on the category  $\mathcal{O}_{\ell+h^\vee}$  of  $\widehat{\mathfrak{g}}$ -modules considered in [38]-[41] and [53]. The main problem is that in this case, modules in  $\mathcal{O}_{\ell+h^\vee}$  are not always

generated by their lowest conformal weight spaces, so  $\mathcal{O}_{\ell+h^\vee}$  is not equivalent to the category of finite-dimensional  $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ -modules. Moreover, KZ equations are no longer sufficient to describe the associativity isomorphisms. It may be possible to obtain an equivalence between  $\mathcal{O}_{\ell+h^\vee}$  and a category of finite-dimensional modules for one of the algebras  $A_N(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ ,  $N \in \mathbb{Z}_+$ , constructed in [6] generalizing Zhu's algebra, and it may be possible to obtain a description of the tensor product in  $\mathcal{O}_{\ell+h^\vee}$  using the description of the space of (logarithmic) intertwining operators among a triple of generalized modules for a vertex operator algebra from [34]. However, the algebras  $A_N(V)$  for a vertex operator algebra  $V$  are typically very difficult to calculate explicitly. Alternatively, it may be possible to use the methods of this paper to describe a tensor category structure on the semisimple subcategory of  $\mathcal{O}_{\ell+h^\vee}$  generated by irreducible  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules. This could possibly be interesting for comparison with the semisimple subquotient of the category of finite-dimensional  $U_q(\mathfrak{g})$ -modules, where  $q$ , as before, is the root of unity corresponding to  $\ell$ . We remark that for the case  $\ell \in \mathbb{Q}_{>-h^\vee} \setminus \mathbb{N}$ , we expect that the category of finitely-generated generalized modules for  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$  should be a braided tensor category, but this does not seem to have been proven yet. In fact, in many cases these categories are not well understood; for instance, although irreducible  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules have been classified for certain rational levels in the case  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$  ([1], [5], [51]), such a classification does not seem to exist in general.

It is also interesting to consider how the methods and results in this paper may generalize to vertex operator algebras associated to other rational conformal field theories. Our analysis of  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules when  $\ell \in \mathbb{N}$  is aided by the particularly simple nature of the KZ equations satisfied by correlation functions corresponding to compositions of intertwining operators. For a general vertex operator algebra  $V$ , Huang showed in [18] that compositions of intertwining operators among  $V$ -modules satisfying the  $C_1$ -cofiniteness condition satisfy systems of regular singular point differential equations. However, these differential equations are not generally explicit and may be of limited use for obtaining an explicit description of the associativity isomorphisms in the tensor category of  $V$ -modules. In situations where we have an explicit description of the abelian category of  $V$ -modules and explicit differential equations for correlation functions, the methods and results of this paper may be generalizable. For example, for Virasoro vertex operator algebras associated to minimal models in rational conformal field theory, the correlation functions satisfy Belavin-Polyakov-Zamolodchikov equations ([3]).

Now we discuss the outline of this paper. Section 2 recalls the affine Lie algebra  $\widehat{\mathfrak{g}}$  and its modules, the vertex operator algebra structure on certain  $\widehat{\mathfrak{g}}$ -modules, such as  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ , and the associated Kac-Moody Lie algebra  $\widetilde{\mathfrak{g}}$ . Moreover, we prove a theorem on the weights of irreducible standard  $\widetilde{\mathfrak{g}}$ -modules. In Section 3, we recall the notions of intertwining operator and  $P(z)$ -intertwining map,  $z \in \mathbb{C}^\times$ , among modules for a vertex operator algebra, as well as the notion of  $P(z)$ -tensor product of modules for a vertex operator algebra. In Section 4, we recall the Zhu's algebra of a vertex operator algebra and the (tensor) equivalence between modules for a suitable vertex operator algebra, such as  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$  when  $\ell \in \mathbb{N}$ , and finite-dimensional modules for its Zhu's algebra. We also recall the classification theorem of [16] and [47] for intertwining operators among certain modules for a vertex operator algebra and use it to describe  $P(z)$ -tensor products among  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules when  $\ell \in \mathbb{N}$ . In Section 5, we recall how products and iterates of intertwining operators among modules for  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$  lead to solutions of the KZ equations, and we construct Drinfeld associators

from solutions to a general one-variable KZ equation at arbitrary level. Finally, in Section 6, we use the associativity of intertwining operators among  $L_{\mathfrak{g}}(\ell, 0)$ -modules to show that under the equivalence of categories given by the Zhu's algebra of  $L_{\mathfrak{g}}(\ell, 0)$ , the associativity isomorphisms in  $L_{\mathfrak{g}}(\ell, 0) - \mathbf{mod}$  correspond to adjoints of Drinfeld associators.

## 2 Affine Lie algebras and their modules

In this section we recall the vertex operator algebras and their modules which come from representations of affine Lie algebras. We also recall some facts about affine Kac-Moody Lie algebras and prove a theorem on the weights of their irreducible standard modules.

### 2.1 Vertex operator algebras from affine Lie algebras

We fix a finite-dimensional simple complex Lie algebra  $\mathfrak{g}$ , with Cartan subalgebra  $\mathfrak{h}$ . The Lie algebra  $\mathfrak{g}$  has a unique up to scale nondegenerate invariant bilinear form  $\langle \cdot, \cdot \rangle$  which remains nondegenerate when restricted to  $\mathfrak{h}$ . Thus  $\langle \cdot, \cdot \rangle$  induces a nondegenerate bilinear form on  $\mathfrak{h}^*$ , and we scale  $\langle \cdot, \cdot \rangle$  so that

$$\langle \alpha, \alpha \rangle = 2$$

when  $\alpha \in \mathfrak{h}^*$  is a long root of  $\mathfrak{g}$ .

We recall some basic facts and notation regarding  $\mathfrak{g}$  (see for example [35] for more details). We recall that the *root lattice*  $Q$  is the  $\mathbb{Z}$ -span of the simple roots  $\{\alpha_i\}_{i=1}^{\text{rank } \mathfrak{g}}$  of  $\mathfrak{g}$ , and we use  $Q_+$  to denote the set of non-negative integral linear combinations of the simple roots. Then  $\mathfrak{h}^*$  has a partial order  $\prec$  given by  $\alpha \prec \beta$  if and only if  $\beta - \alpha \in Q_+$ . The *weight lattice*  $P$  is the set of all  $\lambda \in \mathfrak{h}^*$  such that

$$\frac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \in \mathbb{Z}$$

for each simple root  $\alpha_i$ . The set of *dominant integral weights*  $P_+ \subseteq P$  is the set of weights  $\lambda$  such that

$$\frac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \geq 0$$

for each simple root  $\alpha_i$ . An example of a dominant integral weight is the weight  $\rho$  defined to be half the sum of the positive roots of  $\mathfrak{g}$ . The weight  $\rho$  satisfies  $\frac{2\langle \rho, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1$  for each simple root  $\alpha_i$ .

Every finite-dimensional  $\mathfrak{g}$ -module  $U$  is completely reducible, and the irreducible finite-dimensional  $\mathfrak{g}$ -modules are given by the irreducible highest-weight modules  $L_\lambda$  where  $\lambda \in \mathfrak{h}^*$  is a dominant integral weight. Every  $\mathfrak{g}$ -module  $U$  has a contragredient: the dual space  $U^*$  is a  $\mathfrak{g}$ -module with the action of  $\mathfrak{g}$  given by

$$\langle a \cdot u', u \rangle = -\langle u', a \cdot u \rangle$$

for  $a \in \mathfrak{g}$ ,  $u' \in U^*$ , and  $u \in U$ . Moreover, the tensor product of two  $\mathfrak{g}$ -modules  $U$  and  $V$  is a  $\mathfrak{g}$ -module with the module structure given by

$$a \cdot (u \otimes v) = (a \cdot u) \otimes v + u \otimes (a \cdot v)$$

for  $a \in \mathfrak{g}$ ,  $u \in U$ , and  $v \in V$ .

We also recall the Casimir element  $C = \sum_{i=1}^{\dim \mathfrak{g}} \gamma_i^2 \in U(\mathfrak{g})$ , where  $\{\gamma_i\}_{i=1}^{\dim \mathfrak{g}}$  is an orthonormal basis with respect to the form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . Since  $C$  is central in  $U(\mathfrak{g})$ ,  $C$  acts on finite-dimensional irreducible modules by a scalar. In particular,  $C$  acts on the adjoint representation  $\mathfrak{g}$  by a scalar  $2h^\vee$ , where  $h^\vee$  is the *dual Coxeter number* of  $\mathfrak{g}$ . More generally,  $C$  acts on the irreducible  $\mathfrak{g}$ -module  $L_\lambda$  with highest weight  $\lambda$  by the scalar  $\langle \lambda, \lambda + 2\rho \rangle$ . Since the highest weight of the adjoint representation is the highest root  $\theta$  of  $\mathfrak{g}$ , and since  $\langle \theta, \theta \rangle = 2$ , it follows that

$$\langle \rho, \theta \rangle = h^\vee - 1. \quad (2.1)$$

Because  $\rho \in P_+$ , it follows that  $h^\vee$  is a positive integer.

The affine Lie algebra  $\widehat{\mathfrak{g}}$  is given by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$$

where  $\mathbf{k}$  is central and the remaining brackets are determined by

$$[g \otimes t^m, h \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle g, h \rangle \delta_{m+n,0} \mathbf{k}$$

for  $g, h \in \mathfrak{g}$ ,  $m, n \in \mathbb{Z}$ . We also have the decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_+$$

where

$$\widehat{\mathfrak{g}}_\pm = \coprod_{n \in \pm \mathbb{Z}_+} \mathfrak{g} \otimes t^n, \quad \widehat{\mathfrak{g}}_0 = \mathfrak{g} \otimes t^0 \oplus \mathbb{C}\mathbf{k}.$$

We construct modules for  $\widehat{\mathfrak{g}}$  as follows. Any finite-dimensional  $\mathfrak{g}$ -module  $U$  becomes a  $\widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_+$ -module on which  $\mathfrak{g} \otimes t^n$  acts trivially for  $n > 0$ ,  $\mathfrak{g} \otimes t^0$  acts as  $\mathfrak{g}$ , and  $\mathbf{k}$  as some *level*  $\ell \in \mathbb{C}$ . Then we have the generalized Verma module

$$V_{\widehat{\mathfrak{g}}}(\ell, U) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_+)} U.$$

The generalized Verma module  $V_{\widehat{\mathfrak{g}}}(\ell, U)$  is the linear span of vectors of the form

$$g_1(-n_1) \cdots g_k(-n_k)u$$

where  $g_i \in \mathfrak{g}$ ,  $n_i > 0$ ,  $u \in U$ , and we use the notation  $g(n)$  to denote the action of  $g \otimes t^n$  on a  $\widehat{\mathfrak{g}}$ -module. When  $U$  is an irreducible  $\mathfrak{g}$ -module,  $V_{\widehat{\mathfrak{g}}}(\ell, U)$  has a unique maximal irreducible quotient  $L_{\widehat{\mathfrak{g}}}(\ell, U)$ . If  $U$  is not irreducible, it is still a direct sum of irreducible  $\mathfrak{g}$ -modules, and in this case we use  $L_{\widehat{\mathfrak{g}}}(\ell, U)$  to denote the corresponding direct sum of irreducible  $\widehat{\mathfrak{g}}$ -modules. Note that in any case,  $L_{\widehat{\mathfrak{g}}}(\ell, U)$  is the quotient of  $V_{\widehat{\mathfrak{g}}}(\ell, U)$  by the maximal submodule which does not intersect  $U$ .

**Remark 2.1.** If  $L_\lambda$  is the finite-dimensional irreducible  $\mathfrak{g}$ -module with highest weight  $\lambda \in P_+$ , we will typically use  $V_{\widehat{\mathfrak{g}}}(\ell, \lambda)$  and  $L_{\widehat{\mathfrak{g}}}(\ell, \lambda)$  to denote the  $\widehat{\mathfrak{g}}$ -modules  $V_{\widehat{\mathfrak{g}}}(\ell, L_\lambda)$  and  $L_{\widehat{\mathfrak{g}}}(\ell, L_\lambda)$ , respectively. In particular,  $V_{\widehat{\mathfrak{g}}}(\ell, 0)$  is the generalized Verma module induced from the trivial one-dimensional  $\mathfrak{g}$ -module  $\mathbb{C}\mathbf{1}$ , and  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$  is its irreducible quotient.

When  $\ell \neq -h^\vee$ , the generalized Verma module  $V_{\hat{\mathfrak{g}}}(\ell, 0)$  induced from the one-dimensional  $\mathfrak{g}$ -module  $\mathbb{C}\mathbf{1}$  has the structure of a vertex operator algebra ([16]; see for example [15], [14], and [45] for the definitions of vertex operator algebra and module, and for notation). The vacuum vector of  $V_{\hat{\mathfrak{g}}}(\ell, 0)$  is  $\mathbf{1}$  and the vertex operator map determined by

$$Y(g(-1)\mathbf{1}, x) = g(x) = \sum_{n \in \mathbb{Z}} g(n)x^{-n-1} \quad (2.2)$$

for  $g \in \mathfrak{g}$ . The conformal vector  $\omega$  of  $V_{\hat{\mathfrak{g}}}(\ell, 0)$  is given by

$$\omega = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} \gamma_i(-1)^2 \mathbf{1},$$

where as before  $\{\gamma_i\}_{i=1}^{\dim \mathfrak{g}}$  is an orthonormal basis of  $\mathfrak{g}$ .

It follows from (2.2) and the Jacobi identity for vertex operator algebras that the Virasoro operators  $L(n)$  on  $V_{\hat{\mathfrak{g}}}(\ell, 0)$  for  $n \in \mathbb{Z}$  are given by

$$L(n) = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} \sum_{m \in \mathbb{Z}} \circ \gamma_i(m) \gamma_i(n-m) \circ; \quad (2.3)$$

here the normal ordering notation means

$$\circ g(k)h(l) \circ = \begin{cases} g(k)h(l) & \text{if } k \leq 0 \\ h(l)g(k) & \text{if } k > 0 \end{cases}$$

for  $g, h \in \mathfrak{g}$  and  $k, l \in \mathbb{Z}$ . In particular, we have

$$L(0) = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} \gamma_i(0)^2 + \frac{1}{\ell + h^\vee} \sum_{i=1}^{\dim \mathfrak{g}} \sum_{n>0} \gamma_i(-n) \gamma_i(n) \quad (2.4)$$

and

$$L(-1) = \frac{1}{\ell + h^\vee} \sum_{i=1}^{\dim \mathfrak{g}} \sum_{n \geq 0} \gamma_i(-n-1) \gamma_i(n). \quad (2.5)$$

For any weight  $\lambda \in P_+$  of  $\mathfrak{g}$ , the generalized Verma module  $V_{\hat{\mathfrak{g}}}(\ell, \lambda)$  is a  $V_{\hat{\mathfrak{g}}}(\ell, 0)$ -module with vertex operator map also determined by (2.2). Thus the Virasoro operators  $L(n)$  for  $n \in \mathbb{Z}$  on  $V_{\hat{\mathfrak{g}}}(\ell, \lambda)$  are also given by (2.3). In particular, since  $g(n)$  for  $g \in \mathfrak{g}$  and  $n > 0$  annihilates  $L_\lambda \subseteq V_{\hat{\mathfrak{g}}}(\ell, \lambda)$ , we have from (2.4) that  $L(0)$  acts on  $L_\lambda$  as the scalar

$$h_{\lambda, \ell} = \frac{1}{2(\ell + h^\vee)} \langle \lambda, \lambda + 2\rho \rangle. \quad (2.6)$$

Moreover, one can check that

$$[L(0), g(n)] = -n g(n) \quad (2.7)$$

for any  $g \in \mathfrak{g}$  and  $n \in \mathbb{Z}$ ; this means that the conformal weight gradation of  $V_{\hat{\mathfrak{g}}}(\ell, \lambda)$  is given by

$$\text{wt } g_1(-n_1) \cdots g_k(-n_k)u = n_1 + \dots + n_k + h_{\lambda, \ell}$$

for  $g_i \in \mathfrak{g}$ ,  $n_i > 0$ , and  $u \in L_\lambda$ . The irreducible  $\widehat{\mathfrak{g}}$ -module  $L_{\widehat{\mathfrak{g}}}(\ell, \lambda)$  is also a  $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ -zero module, and in fact every irreducible  $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module is isomorphic to  $L_{\widehat{\mathfrak{g}}}(\ell, \lambda)$  for some  $\lambda \in P_+$ .

The irreducible  $\widehat{\mathfrak{g}}$ -module  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$  is also a vertex operator algebra with vertex operator algebra structure induced from  $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ . We shall be particularly interested in the case  $\ell \in \mathbb{N}$ . Then the category of  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules consists of all standard level  $\ell$   $\widehat{\mathfrak{g}}$ -modules ([16]). In particular, all  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules are completely reducible, and the irreducible  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules are given by  $L_{\widehat{\mathfrak{g}}}(\ell, \lambda)$  where  $\lambda \in P_+$  satisfies  $\langle \lambda, \theta \rangle \leq \ell$ .

## 2.2 The Kac-Moody Lie algebra $\widetilde{\mathfrak{g}}$

Here we recall the affine Kac-Moody Lie algebra  $\widetilde{\mathfrak{g}}$  and some of its properties that we shall need later; for more details and proofs, see references such as [44], [36], [49], and [4]. Let us extend the algebra  $\widehat{\mathfrak{g}}$  to the Kac-Moody Lie algebra

$$\widetilde{\mathfrak{g}} = \widehat{\mathfrak{g}} \rtimes \mathbb{C}d$$

where  $[d, k] = 0$  and

$$[d, g \otimes t^n] = n g \otimes t^n \quad (2.8)$$

for  $g \in \mathfrak{g}$  and  $n \in \mathbb{Z}$ . Recalling the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ ,

$$\mathfrak{H} = \mathfrak{h} \oplus \mathbb{C}k \oplus \mathbb{C}d$$

is a Cartan subalgebra for  $\widetilde{\mathfrak{g}}$ . Then  $\mathfrak{H}^*$  has a basis  $\{\alpha_i\}_{i=1}^{\text{rank } \mathfrak{g}} \cup \{k', d'\}$  where  $k'$  and  $d'$  are dual to  $k$  and  $d$ , respectively. The remaining simple root of  $\widetilde{\mathfrak{g}}$  is given by

$$\alpha_0 = -\theta + d',$$

where as previously  $\theta$  is the longest root of  $\mathfrak{g}$ . We use  $\Delta$  to denote the set of roots of  $\widetilde{\mathfrak{g}}$  and  $\Delta_\pm$  to denote the sets of positive and negative roots of  $\widetilde{\mathfrak{g}}$ , respectively. We use  $\widehat{Q}$  to denote the root lattice of  $\widetilde{\mathfrak{g}}$ , the integer span of the roots of  $\widetilde{\mathfrak{g}}$ . Note that

$$\widehat{Q} = Q \oplus \mathbb{Z}d'.$$

We use  $\widehat{Q}_+$  to denote the non-negative integral span of the simple roots of  $\widetilde{\mathfrak{g}}$ . Then the partial order on  $\mathfrak{h}^*$  extends to a partial order  $\prec$  on  $\mathfrak{H}^*$  given by  $\alpha \prec \beta$  if and only if  $\beta - \alpha \in \widehat{Q}_+$ .

We extend the bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{h}^*$  to a nondegenerate symmetric form on  $\mathfrak{H}^*$  by setting

$$\langle k', d' \rangle = 1 \quad (2.9)$$

and

$$\langle k', k' \rangle = \langle d', d' \rangle = \langle k', \alpha_i \rangle = \langle d', \alpha_i \rangle = 0$$

for  $i \geq 1$ . Now we use  $\widehat{P}$  to denote the *weights* of  $\widetilde{\mathfrak{g}}$ , the elements  $\Lambda$  of  $\mathfrak{H}^*$  such that

$$\frac{2\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \in \mathbb{Z}$$

for all  $0 \leq i \leq \text{rank } \mathfrak{g}$ . Note that

$$\widehat{P} = P \oplus \mathbb{Z}\mathbf{k}' \oplus \mathbb{C}\mathbf{d}'.$$

We say that a weight  $\Lambda \in \widehat{P}$  is *dominant integral* if

$$\frac{2\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \in \mathbb{N}$$

for all  $0 \leq i \leq \text{rank } \mathfrak{g}$ ; we use  $\widehat{P}_+$  to denote the dominant integral weights of  $\widetilde{\mathfrak{g}}$ . Note that  $\Lambda = \lambda + \ell\mathbf{k}' + h\mathbf{d}'$ , where  $\lambda \in P$  and  $\ell \in \mathbb{Z}$ , is dominant integral if and only if  $\ell \in \mathbb{N}$  and  $\lambda$  is a dominant integral weight of  $\mathfrak{g}$  such that  $\langle \lambda, \theta \rangle \leq \ell$ .

Recalling the dominant integral weight  $\rho$  of  $\mathfrak{g}$  and the dual Coxeter number  $h^\vee$ , we have:

**Proposition 2.2.** *If we set  $\widehat{\rho} = \rho + h^\vee\mathbf{k}'$ , then  $\frac{2\langle \widehat{\rho}, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1$  for all simple roots  $\alpha_i$  of  $\widetilde{\mathfrak{g}}$ .*

*Proof.* It is clear that  $\frac{2\langle \widehat{\rho}, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1$  for  $i \geq 1$  since  $\rho$  has this property. Since  $\langle \alpha_0, \alpha_0 \rangle = 2$ , we just need to check  $\langle \widehat{\rho}, \alpha_0 \rangle = 1$ . In fact,

$$\langle \widehat{\rho}, \alpha_0 \rangle = \langle \rho + h^\vee\mathbf{k}', -\theta + \mathbf{d}' \rangle = -(h^\vee - 1) + h^\vee = 1$$

by (2.1) and (2.9). □

We now recall the Weyl group of  $\widetilde{\mathfrak{g}}$ . For each  $0 \leq i \leq \text{rank } \mathfrak{g}$ , we define the simple reflection  $r_i$  on  $\mathfrak{H}^*$  by

$$r_i(\alpha) = \alpha - \frac{2\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$$

for  $\alpha \in \mathfrak{H}^*$ . Then the Weyl group  $W$  is the group of isometries of  $\mathfrak{H}^*$  generated by the reflections  $r_i$ . For an element  $w \in W$ , the *length*  $l(w)$  is the minimal number of generators in any expression of  $w$  as a product of the generators  $r_i$ . Here we recall some standard facts about the Weyl group of  $\widetilde{\mathfrak{g}}$  that we shall need:

**Proposition 2.3.** *For  $w \in W$ , let  $\Phi_w \subseteq \Delta_+$  denote the positive roots sent by  $w$  to negative roots. Then for any  $w \in W$ ,  $l(w) = |\Phi_w|$  and  $\widehat{\rho} - w(\widehat{\rho}) = \sum_{\alpha \in \Phi_w} \alpha$ .*

**Proposition 2.4.** *The Weyl group  $W$  preserves the set of weights  $\widehat{P}$ , and for any  $\Lambda \in \widehat{P}$ , there is some  $w \in W$  such that  $w(\Lambda) \in \widehat{P}_+$ .*

Now we consider  $\widetilde{\mathfrak{g}}$ -modules. For each  $\Lambda \in \widehat{P}_+$ , there is a unique irreducible standard  $\widetilde{\mathfrak{g}}$ -module  $L(\Lambda)$  with highest weight  $\Lambda$ . Note that by (2.7) and (2.8), for any  $\lambda \in P_+$  and  $\ell \in \mathbb{C}$ , the  $\widehat{\mathfrak{g}}$ -modules  $V_{\widehat{\mathfrak{g}}}(\ell, \lambda)$  and  $L_{\widehat{\mathfrak{g}}}(\ell, \lambda)$  become  $\widetilde{\mathfrak{g}}$ -modules on which  $\mathbf{d}$  acts as  $-L(0)$ . In particular, when  $\ell \in \mathbb{N}$  and  $\lambda \in P_+$  satisfies  $\langle \lambda, \theta \rangle \leq \ell$ ,  $L_{\widehat{\mathfrak{g}}}(\ell, \lambda) \cong L(\Lambda)$  where

$$\Lambda = \lambda + \ell\mathbf{k}' - h_{\lambda, \ell}\mathbf{d}' \in \widehat{P}_+.$$

For a dominant integral weight  $\Lambda \in \widehat{P}_+$ , the weights of  $L(\Lambda)$  lie in  $\Lambda - \widehat{Q}_+$ . Moreover, the Weyl group  $W$  preserves the weights of any standard  $\widetilde{\mathfrak{g}}$ -module. Thus we obtain:

**Proposition 2.5.** *If  $\Lambda'$  is a weight of  $L(\Lambda)$  and  $w \in W$ , then  $w(\Lambda') \prec \Lambda$ .*

We will need the following theorem on the weights of irreducible  $\tilde{\mathfrak{g}}$ -modules. Suppose that  $\Lambda = \lambda + \ell \mathbf{k}' - h_{\lambda, \ell} \mathbf{d}'$  is a dominant integral weight of  $\tilde{\mathfrak{g}}$ , where  $\ell \in \mathbb{N}$  and  $\lambda \in P_+$  satisfies  $\langle \lambda, \theta \rangle \leq \ell$ .

**Theorem 2.6.** *If  $\Lambda' = \lambda' + \ell \mathbf{k}' - (h_{\lambda', \ell} - m) \mathbf{d}'$  is a weight of  $L(\Lambda)$  such that  $\lambda' \in P_+$  and  $m \in \mathbb{N}$ , then  $\Lambda' = \Lambda$ .*

*Proof.* By Proposition 2.4 there is a weight  $\tilde{\Lambda} \in \hat{P}$  and an element  $w \in W$  such that

$$w^{-1}(\Lambda' + \hat{\rho}) = \tilde{\Lambda} + \hat{\rho} \in \hat{P}_+. \quad (2.10)$$

**Lemma 2.7.** *We have  $\tilde{\Lambda} = \tilde{\lambda} + \ell \mathbf{k}' - (h_{\tilde{\lambda}, \ell} - m) \mathbf{d}'$  for some  $\tilde{\lambda} \in P$  which satisfies  $\tilde{\lambda} + \rho \in P_+$  and  $\langle \tilde{\lambda}, \theta \rangle \leq \ell + 1$ .*

*Proof.* Certainly we can write  $\tilde{\Lambda} = \tilde{\lambda} + \tilde{\ell} \mathbf{k}' - h \mathbf{d}'$  for some  $\tilde{\lambda} \in P$ ,  $\tilde{\ell} \in \mathbb{Z}$ , and  $h \in \mathbb{C}$ . First we observe that for any weight  $\Lambda' \in \hat{P}$  and any simple generator  $r_i$  of  $W$ , the coefficient of  $\mathbf{k}'$  in

$$r_i(\Lambda') = \Lambda' - \frac{2\langle \Lambda', \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$$

is the same as the coefficient of  $\mathbf{k}'$  in  $\Lambda'$  because  $\alpha_i \in Q + \mathbb{Z} \mathbf{d}'$ . Thus  $\tilde{\ell} = \ell$ . Also, the fact that  $\tilde{\Lambda} + \hat{\rho} \in \hat{P}_+$  implies that  $\tilde{\lambda} + \rho \in P_+$  and

$$\langle \lambda + \rho, \theta \rangle \leq \ell + h^\vee.$$

By (2.1), this implies  $\langle \tilde{\lambda}, \theta \rangle \leq \ell + 1$ .

It remains to show that  $h = h_{\tilde{\lambda}, \ell} - m$ . For this, we use the fact that

$$\begin{aligned} \langle \tilde{\Lambda}, \tilde{\Lambda} + 2\hat{\rho} \rangle &= \langle w^{-1}(\Lambda' + \hat{\rho}) - \hat{\rho}, w^{-1}(\Lambda' + \hat{\rho}) + \hat{\rho} \rangle \\ &= \langle w^{-1}(\Lambda' + \hat{\rho}), w^{-1}(\Lambda' + \hat{\rho}) \rangle - \langle \hat{\rho}, \hat{\rho} \rangle \\ &= \langle \Lambda', \Lambda' + 2\hat{\rho} \rangle \end{aligned}$$

because  $w^{-1}$  is an isometry. Now,

$$\begin{aligned} \langle \Lambda', \Lambda' + 2\hat{\rho} \rangle &= \langle \lambda' + \ell \mathbf{k}' - (h_{\lambda', \ell} - m) \mathbf{d}', \lambda' + 2\rho + (\ell + 2h^\vee) \mathbf{k}' - (h_{\lambda', \ell} - m) \mathbf{d}' \rangle \\ &= \langle \lambda', \lambda' + 2\rho \rangle - 2(\ell + h^\vee)(h_{\lambda', \ell} - m) = 2m(\ell + h^\vee) \end{aligned}$$

by the definition of  $h_{\lambda', \ell}$ . Thus

$$\langle \tilde{\Lambda}, \tilde{\Lambda} + 2\hat{\rho} \rangle = \langle \tilde{\lambda}, \tilde{\lambda} + 2\rho \rangle - 2(\ell + h^\vee)h = 2m(\ell + h^\vee)$$

likewise, so

$$h = \frac{1}{2(\ell + h^\vee)} (\langle \tilde{\lambda}, \tilde{\lambda} + 2\rho \rangle - 2m(\ell + h^\vee)) = h_{\tilde{\lambda}, \ell} - m,$$

as desired. □

Continuing with the proof of the theorem, by Propositions 2.3 and 2.5 we have

$$\tilde{\Lambda} = w^{-1}(\Lambda') + w^{-1}(\hat{\rho}) - \hat{\rho} \prec \Lambda.$$

Set  $N = h_{\tilde{\lambda}, \ell} - h_{\lambda, \ell}$ . Since we know

$$\Lambda - \tilde{\Lambda} = \lambda - \tilde{\lambda} + (h_{\tilde{\lambda}, \ell} - h_{\lambda, \ell} - m)\mathbf{d}' = \lambda - \tilde{\lambda} + (N - m)\theta + (N - m)\alpha_0 \in \hat{Q}_+, \quad (2.11)$$

it follows that  $N - m \in \mathbb{N}$  (so that  $N \in \mathbb{N}$  as well) and  $\tilde{\lambda} \prec \lambda + (N - m)\theta \prec \lambda + N\theta$ . The following lemma then implies that  $N = m = 0$ :

**Lemma 2.8.** *Suppose  $\lambda, \tilde{\lambda}$  are weights of  $\mathfrak{g}$  such that  $\lambda + \tilde{\lambda} + \rho \in P_+$ ,  $\langle \lambda + \tilde{\lambda}, \theta \rangle \leq 2\ell + 1$ , and  $\tilde{\lambda} \prec \lambda + N\theta$  for some  $N \in \mathbb{N}$ . Then  $h_{\tilde{\lambda}, \ell} - h_{\lambda, \ell} < N$  if  $N > 0$ .*

*Proof.* Because  $\tilde{\lambda} - \lambda \prec N\theta$ , we have  $\langle \tilde{\lambda} - \lambda, \mu \rangle \leq N\langle \theta, \mu \rangle$  whenever  $\mu \in P_+$ . In particular,

$$\langle \tilde{\lambda} - \lambda, \rho \rangle \leq N\langle \rho, \theta \rangle = N(h^\vee - 1)$$

by (2.1) and

$$\langle \tilde{\lambda} - \lambda, \lambda + \tilde{\lambda} + \rho \rangle \leq N\langle \lambda + \tilde{\lambda} + \rho, \theta \rangle \leq N(2\ell + h^\vee).$$

Then we have

$$\begin{aligned} h_{\tilde{\lambda}, \ell} - h_{\lambda, \ell} &= \frac{1}{2(\ell + h^\vee)} \left( \langle \tilde{\lambda}, \tilde{\lambda} + 2\rho \rangle - \langle \lambda, \lambda + 2\rho \rangle \right) \\ &= \frac{1}{2(\ell + h^\vee)} \langle \tilde{\lambda} - \lambda, \lambda + \tilde{\lambda} + 2\rho \rangle \\ &\leq \frac{2(\ell + h^\vee) - 1}{2(\ell + h^\vee)} N. \end{aligned}$$

Thus  $h_{\tilde{\lambda}, \ell} - h_{\lambda, \ell} < N$  when  $N > 0$ . □

We have now proved that  $\tilde{\lambda} \prec \lambda$  and  $h_{\tilde{\lambda}, \ell} = h_{\lambda, \ell}$ . Thus  $\lambda = \tilde{\lambda} + \alpha$  for some  $\alpha \in Q_+$  and

$$\langle \tilde{\lambda}, \tilde{\lambda} + 2\rho \rangle = \langle \tilde{\lambda} + \alpha, \tilde{\lambda} + \alpha + 2\rho \rangle,$$

or

$$\langle \alpha, \alpha \rangle + 2\langle \tilde{\lambda} + \rho, \alpha \rangle = 0.$$

Since  $\tilde{\lambda} + \rho \in P_+$  by Lemma 2.7, this means  $\langle \alpha, \alpha \rangle = \langle \tilde{\lambda} + \rho, \alpha \rangle = 0$ . Thus we must have  $\alpha = 0$  and  $\tilde{\lambda} = \lambda$ . Then by (2.11) we have  $\tilde{\Lambda} = \Lambda$ .

It now follows from (2.10) that  $w^{-1}(\Lambda' + \hat{\rho}) = \Lambda + \hat{\rho}$ , or

$$\Lambda' = w(\Lambda + \hat{\rho}) - \hat{\rho}.$$

To conclude the proof of the theorem, we must show that  $w = 1$ . In fact, when  $w \neq 1$ , Proposition 2.3 implies that

$$w^{-1}(\Lambda') = \Lambda + \hat{\rho} - w^{-1}(\hat{\rho}) = \Lambda + \sum_{\alpha \in \Phi_{w^{-1}}} \alpha \neq \Lambda.$$

Since by assumption  $\Lambda'$  is a weight of  $L(\Lambda)$ , this contradicts Proposition 2.5; thus  $w = 1$  and  $\Lambda' = \Lambda$ . This completes the proof of the theorem. □

### 3 Intertwining operators and tensor products of modules for a vertex operator algebra

In this section we recall the notion of tensor product of modules for a vertex operator algebra from [22], [28]. We first recall the notions of intertwining operator and  $P(z)$ -intertwining map, for  $z \in \mathbb{C}^\times$ , among a triple of modules for a vertex operator algebra.

#### 3.1 Intertwining operators and $P(z)$ -intertwining maps

In general for a vector space  $V$ , we use  $V\{x\}$  to denote the space of formal series

$$V\{x\} = \left\{ \sum_{n \in \mathbb{C}} v_n x^n, v_n \in V \right\}.$$

Now suppose that  $V$  is a vertex operator algebra; we recall the notion of intertwining operator among a triple of  $V$ -modules (see [14] or [22]):

**Definition 3.1.** Suppose  $W_1$ ,  $W_2$  and  $W_3$  are  $V$ -modules. An *intertwining operator* of type  $\binom{W_3}{W_1 W_2}$  is a linear map

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\rightarrow W_3\{x\}, \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} (w_{(1)})_n w_{(2)} x^{-n-1} \in W_3\{x\} \end{aligned}$$

satisfying the following conditions:

1. *Lower truncation:* For any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $n \in \mathbb{C}$ ,

$$(w_{(1)})_{(n+m)} w_{(2)} = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large.} \quad (3.1)$$

2. The *Jacobi identity*:

$$\begin{aligned} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_{W_3}(v, x_1) \mathcal{Y}(w_{(1)}, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_{(1)}, x_2) Y_{W_2}(v, x_1) \\ = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y_{W_1}(v, x_0) w_{(1)}, x_2) \end{aligned} \quad (3.2)$$

for  $v \in V$  and  $w_{(1)} \in W_1$ .

3. The  *$L(-1)$ -derivative property*: for any  $w_{(1)} \in W_1$ ,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx} \mathcal{Y}(w_{(1)}, x). \quad (3.3)$$

**Remark 3.2.** We use  $\mathcal{V}_{W_1 W_2}^{W_3}$  to denote the vector space of intertwining operators of type  $\binom{W_3}{W_1 W_2}$ ; the dimension of  $\mathcal{V}_{W_1 W_2}^{W_3}$  is the corresponding *fusion rule*  $\mathcal{N}_{W_1 W_2}^{W_3}$ .

**Remark 3.3.** The vertex operator  $Y$  of  $V$  is an intertwining operator of type  $\binom{V}{VV}$ , and for a  $V$ -module  $W$ , the vertex operator  $Y_W$  is an intertwining operator of type  $\binom{W}{VW}$ .

Taking the coefficient of  $x_0^{-1}$  in the Jacobi identity (3.2) yields the *commutator formula*

$$Y_{W_3}(v, x_1)\mathcal{Y}(w_{(1)}, x_2) - \mathcal{Y}(w_{(1)}, x_2)Y_{W_2}(v, x_1) = \text{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_{W_1}(v, x_0)w_{(1)}, x_2), \quad (3.4)$$

where the formal residue notation  $\text{Res}_{x_0}$  denotes the coefficient of  $x_0^{-1}$ . Similarly, taking the coefficient of  $x_1^{-1}$  in (3.2) and then the coefficient of  $x_0^{-n-1}$  for any  $n \in \mathbb{Z}$  yields the *iterate formula*

$$\mathcal{Y}(v_n w_{(1)}, x_2) = \text{Res}_{x_1} \left( (x_1 - x_2)^n Y_{W_3}(v, x_1) \mathcal{Y}(w_{(1)}, x_2) - (-x_2 + x_1)^n \mathcal{Y}(w_{(1)}, x_2) Y_{W_2}(v, x_1) \right). \quad (3.5)$$

Together, the commutator and iterate formulas are equivalent to the Jacobi identity (see [45] for the special case that  $\mathcal{Y} = Y_W$  for a  $V$ -module  $W$ ).

We now recall from [22], [28] the notion of  $P(z)$ -intertwining map for  $z \in \mathbb{C}^\times$ . First, if

$$W = \coprod_{n \in \mathbb{C}} W_{(n)}$$

is a  $\mathbb{C}$ -graded vector space (such as a  $V$ -module), then the *algebraic completion* of  $W$  is the vector space

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{(n)}.$$

For any  $n \in \mathbb{C}$ , we use  $\pi_n$  to denote the canonical projection  $\overline{W} \rightarrow W_{(n)}$ . Recall that the graded dual of a graded vector space  $W$  is given by

$$W' = \prod_{n \in \mathbb{C}} W_{(n)}^*;$$

it is easy to see that  $\overline{W} = (W')^*$ .

**Definition 3.4.** Suppose  $W_1$ ,  $W_2$ , and  $W_3$  are  $V$ -modules and  $z \in \mathbb{C}^\times$ . A  $P(z)$ -intertwining map of type  $\binom{W_3}{W_1 W_2}$  is a linear map

$$I : W_1 \otimes W_2 \rightarrow \overline{W_3}$$

satisfying the following conditions:

1. *Lower truncation:* For any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $n \in \mathbb{C}$ ,

$$\pi_{n-m}(I(w_{(1)} \otimes w_{(2)})) = 0 \quad \text{for } m \in \mathbb{N} \text{ sufficiently large.} \quad (3.6)$$

2. The *Jacobi identity*:

$$\begin{aligned}
x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) Y_{W_3}(v, x_1) I(w_{(1)} \otimes w_{(2)}) - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) I(w_{(1)} \otimes Y_{W_2}(v, x_1) w_{(2)}) \\
= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) I(Y_{W_1}(v, x_0) w_{(1)} \otimes w_{(2)})
\end{aligned} \tag{3.7}$$

for  $v \in V$ ,  $w_{(1)} \in W_1$ , and  $w_{(2)} \in W_2$ .

**Remark 3.5.** We use  $\mathcal{M}[P(z)]_{W_1 W_2}^{W_3}$ , or simply  $\mathcal{M}_{W_1 W_2}^{W_3}$  if  $z$  is clear, to denote the space of  $P(z)$ -intertwining maps of type  $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ .

The definitions suggest that a  $P(z)$ -intertwining map is essentially an intertwining operator with the formal variable  $x$  specialized to the complex number  $z$ . To make such a substitution precise, however, we must fix a branch of logarithm. We use  $\log z$  to denote the following branch of logarithm with a branch cut along the positive real axis:

$$\log z = \log |z| + i \arg z,$$

where  $0 \leq \arg z < 2\pi$ . Then we define

$$\ell_p(z) = \log z + 2\pi i p$$

for any  $p \in \mathbb{Z}$ . Then from [28], we have

**Proposition 3.6.** *For any  $p \in \mathbb{Z}$ , there is a linear isomorphism  $\mathcal{Y}_{W_1 W_2}^{W_3} \rightarrow \mathcal{M}_{W_1 W_2}^{W_3}$  given by*

$$\mathcal{Y} \mapsto I_{\mathcal{Y}, p},$$

where

$$I_{\mathcal{Y}, p}(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, e^{\ell_p(z)})$$

for  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ . The inverse is given by

$$I \mapsto \mathcal{Y}_{I, p}$$

where

$$\mathcal{Y}_{I, p}(w_{(1)}, x) w_{(2)} = \left( \frac{e^{\ell_p(z)}}{x} \right)^{-L(0)} I \left( \left( \frac{e^{\ell_p(z)}}{x} \right)^{L(0)} w_{(1)} \otimes \left( \frac{e^{\ell_p(z)}}{x} \right)^{L(0)} w_{(2)} \right)$$

for  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ .

**Remark 3.7.** From Proposition 3.6, the dimension of  $\mathcal{M}[P(z)]_{W_1 W_2}^{W_3}$  for any  $z \in \mathbb{C}^\times$  is also the fusion rule  $\mathcal{N}_{W_1 W_2}^{W_3}$

### 3.2 $P(z)$ -tensor products

We now recall from [22] and [28] the definition of a  $P(z)$ -tensor product of  $V$ -modules  $W_1$  and  $W_2$  using a universal property:

**Definition 3.8.** For  $z \in \mathbb{C}^\times$ , a  $P(z)$ -*tensor product* of  $V$ -modules  $W_1$  and  $W_2$  is a  $V$ -module  $W_1 \boxtimes_{P(z)} W_2$  equipped with a  $P(z)$ -intertwining map

$$\boxtimes_{P(z)} : W_1 \otimes W_2 \rightarrow \overline{W_1 \boxtimes_{P(z)} W_2}$$

such that if  $W_3$  is any  $V$ -module and  $I$  is any  $P(z)$ -intertwining map of type  $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ , then there is a unique  $V$ -module homomorphism

$$\eta : W_1 \boxtimes_{P(z)} W_2 \rightarrow W_3$$

satisfying

$$\overline{\eta} \circ \boxtimes_{P(z)} = I.$$

**Remark 3.9.** We typically use the notation

$$w_{(1)} \boxtimes_{P(z)} w_{(2)} = \boxtimes_{P(z)}(w_{(1)} \otimes w_{(2)})$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ .

If  $V$  is a suitable vertex operator algebra, then  $P(z)$ -tensor products of  $V$ -modules always exist. In particular,  $P(z)$ -tensor products exist when  $V$  is finitely reductive in the sense of [28]:

**Definition 3.10.** A vertex operator algebra  $V$  is *finitely reductive* if:

1. Every  $V$ -module is completely reducible.
2. There are finitely many equivalence classes of irreducible  $V$ -modules.
3. All fusion rules for triples of  $V$ -modules are finite.

**Example 3.11.** When  $\ell \in \mathbb{N}$ , the vertex operator algebra  $L_{\hat{\mathfrak{g}}}(\ell, 0)$  is finitely reductive ([16]).

Suppose tensor products of  $V$ -modules exist. In order to give the category  $V - \mathbf{mod}$  a tensor category structure, we must choose a specific tensor product bifunctor, say  $\boxtimes_{P(1)}$ . Under suitable additional conditions,  $V - \mathbf{mod}$  with the tensor product  $\boxtimes_{P(1)}$  becomes a braided tensor category (see [22]-[24], [17] or [26]-[33]). We will discuss the associativity isomorphisms for this tensor category below.

## 4 The Zhu's algebra of a vertex operator algebra and its applications

In this section we recall from [54] that the category of modules for a suitable vertex operator algebra  $V$  is equivalent to the category of finite-dimensional modules for the Zhu's algebra  $A(V)$  of  $V$ . We also recall the connection given in [16], [47] between  $A(V)$  and intertwining operators among  $V$ -modules, and apply these results to the vertex operator algebra  $L_{\hat{\mathfrak{g}}}(\ell, 0)$  when  $\ell \in \mathbb{N}$ .

## 4.1 Zhu's algebra and an equivalence of categories

Here we recall the Zhu's algebra of a vertex operator algebra  $V$  from [54]; see also [16]. The vertex operator algebra  $V$  has a product  $*$  given by

$$u * v = \text{Res}_x x^{-1} Y((1+x)^{L(0)} u, x) v$$

for  $u, v \in V$ . We also define the subspace  $O(V) \subseteq V$  as the linear span of elements

$$\text{Res}_x x^{-2} Y((1+x)^{L(0)} u, x) v$$

for  $u, v \in V$ . Then  $*$  is a well-defined product on the quotient  $A(V) = V/O(V)$ , and in fact  $(A(V), *)$  is an associative algebra with unit  $1 + O(V)$  called the *Zhu's algebra* of  $V$ .

If  $W$  is a  $V$ -module, then there is a left action of  $V$  on  $W$  defined by

$$v * w = \text{Res}_x x^{-1} Y((1+x)^{L(0)} v, x) w$$

for  $v \in V, w \in W$ , and a right action defined by

$$w * v = \text{Res}_x x^{-1} Y((1+x)^{L(0)-1} v, x) w$$

for  $v \in V, w \in W$ . If we define the subspace  $O(W) \subseteq W$  as the linear span of elements of the form

$$\text{Res}_x x^{-2} Y((1+x)^{L(0)} v, x) w$$

for  $v \in V, w \in W$ , then these left and right actions define an  $A(V)$ -bimodule structure on the quotient  $A(W) = W/O(W)$ .

Now, following [19], for a  $V$ -module  $W$  we define the *top level*  $T(W)$  to be the subspace

$$T(W) = \{w \in W \mid v_n w = 0, v \in V \text{ homogeneous, wt } v - n - 1 < 0\}.$$

Then  $T(W)$  is a (left)  $A(V)$ -module with action defined by

$$(v + O(V)) \cdot w = o(v)w$$

for  $v \in V, w \in T(W)$ ; here  $o(v) = v_{\text{wt } v - 1}$  if  $v$  is homogeneous and we extend linearly to define  $o(v)$  for non-homogeneous  $v$ . The correspondence  $W \rightarrow T(W)$  defines a functor

$$T : V\text{-mod} \rightarrow A(V)\text{-mod},$$

in which a  $V$ -module homomorphism  $f : W_1 \rightarrow W_2$  corresponds to  $T(f) = f|_{T(W_1)} : T(W_1) \rightarrow T(W_2)$ . Note that the image of the restriction of a  $V$ -homomorphism to the top level of  $W_1$  is indeed contained in the top level of  $W_2$ .

**Remark 4.1.** If  $W$  is an irreducible  $V$ -module,  $T(W)$  is simply the lowest conformal weight space of  $W$ , and if  $W$  is semisimple,  $T(W)$  is the direct sum of the lowest weight spaces of the irreducible components of  $W$ . In general, however, determining  $T(W)$  is a subtle problem.

When  $V$  is a suitable vertex operator algebra, we want  $T$  to give an equivalence of categories between the category of  $V$ -modules and the category of finite-dimensional (left)  $A(V)$ -modules. Thus we need to construct a functor  $S$  from the category  $\mathbf{C}^{fin}(A(V))$  of finite-dimensional (left)  $A(V)$ -modules to the category of  $V$ -modules. Starting from a finite-dimensional  $A(V)$ -module  $U$ , there are several constructions of a suitable  $V$ -module  $S(U)$  (see [54], [47], and [19]), all of which involve a kind of induced module construction. Here we sketch the construction from [19].

We consider the affinization  $V[t, t^{-1}] = V \otimes \mathbb{C}[t, t^{-1}]$  and its tensor algebra  $\mathcal{T}(V[t, t^{-1}])$ . Then given a finite-dimensional  $A(V)$ -module  $U$ , the vector space  $\mathcal{T}(V[t, t^{-1}]) \otimes U$  is a left  $\mathcal{T}(V[t, t^{-1}])$ -module in the obvious way. For  $v \in V$  and  $n \in \mathbb{Z}$ , we use  $v(n)$  to denote the action of  $v \otimes t^n$  on  $\mathcal{T}(V[t, t^{-1}]) \otimes U$ , and we set

$$Y_t(v, x) = \sum_{n \in \mathbb{Z}} v(n) x^{-n-1}.$$

Now we let  $\mathcal{I}$  denote the submodule of  $\mathcal{T}(V[t, t^{-1}]) \otimes U$  generated by suitable elements so that on the quotient  $S_1(U) = (\mathcal{T}(V[t, t^{-1}]) \otimes U) / \mathcal{I}$ ,  $v(n)$  for  $v \in V$  homogeneous and  $\text{wt } v - n - 1 < 0$  acts trivially on  $U$ ,  $v(\text{wt } v - 1)$  for  $v \in V$  homogeneous acts on  $U$  as  $v + O(V)$  acts, and a commutator formula for the operators  $Y_t(v, x)$  holds. Next, we define  $\mathcal{J}$  to be the submodule of  $S_1(U)$  generated by suitable elements so that on the quotient  $S(U) = S_1(U) / \mathcal{J}$ , an  $L(-1)$ -derivative property and an iterate formula hold for the operators  $Y_t(v, x)$ . Then  $S(U)$  equipped with the vertex operator  $Y(v, x) = Y_t(v, x)$  for  $v \in V$  satisfies all the axioms for a  $V$ -module except that it does not necessarily have a conformal weight grading by  $L(0)$ -eigenvalues satisfying the usual grading restriction conditions (recall the definition of module for a vertex operator algebra from [15], [14], or [45]).

However,  $S(U)$  does admit an  $\mathbb{N}$ -grading  $S(U) = \coprod_{n \geq 0} S(U)(n)$  determined by the properties  $S(U)(0) = U$  and for any homogeneous  $v \in V$  and  $n \in \mathbb{Z}$ ,  $v_n = v(n)$  is an operator of degree  $\text{wt } v - n - 1$ . This means  $S(U)$  is an  $\mathbb{N}$ -gradable weak  $V$ -module in the sense of [19] (such modules are called simply  $V$ -modules in [54], but we reserve the term  $V$ -module for modules with a conformal weight grading satisfying the grading restriction conditions). Note that  $U$  is contained in the top level of  $S(U)$ .

**Remark 4.2.** Note that  $U$  generates  $S(U)$ , so when  $S(U)$  is semisimple,  $T(S(U)) = U$ . However,  $S(U)$  is not generally semisimple, and  $T(S(U))$  does not generally equal  $U$ .

The correspondence  $U \rightarrow S(U)$  defines a functor from  $\mathbf{C}^{fin}(A(V))$  to the category of  $\mathbb{N}$ -gradable weak  $V$ -modules. The fact that an  $A(V)$ -module homomorphism  $f : U_1 \rightarrow U_2$  extends to a unique  $V$ -homomorphism  $S(f) : S(U_1) \rightarrow S(U_2)$  amounts to an appropriate universal property satisfied by the induced module  $S(U_1)$ . Now, if every  $\mathbb{N}$ -gradable weak  $V$ -module is a direct sum of irreducible  $V$ -modules, then for any finite-dimensional  $A(V)$ -module  $U$ ,  $S(U)$  is a direct sum of finitely many irreducible  $V$ -modules since it is generated by the finite-dimensional space  $U$ . In this case,  $S$  is a functor from  $\mathbf{C}^{fin}(A(V))$  to  $V\text{-}\mathbf{mod}$ . Then from [54] and [19], we have:

**Theorem 4.3.** *If every  $\mathbb{N}$ -gradable weak  $V$ -module is a direct sum of irreducible  $V$ -modules, then the functors*

$$T : V\text{-}\mathbf{mod} \rightarrow \mathbf{C}^{fin}(A(V))$$

and

$$S : \mathbf{C}^{fin}(A(V)) \rightarrow V - \mathbf{mod}$$

are equivalences of categories. More specifically,  $T \circ S = 1_{\mathbf{C}^{fin}(A(V))}$  and  $S \circ T$  is naturally isomorphic to  $1_{V - \mathbf{mod}}$ .

## 4.2 Zhu's algebra and intertwining operators

Now we consider intertwining operators among  $V$ -modules. If a  $V$ -module  $W$  is indecomposable, its conformal weights lie in  $h + \mathbb{N}$  for some  $h \in \mathbb{C}$ . Thus suppose  $W_i$  for  $i = 1, 2, 3$  are indecomposable  $V$ -modules whose lowest weights are  $h_i \in \mathbb{C}$ . Then for any intertwining operator  $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$ , we can write ([14])

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{Z}} o_n^{\mathcal{Y}}(w_{(1)} \otimes w_{(2)})x^{h_3 - h_1 - h_2 - n - 1}$$

for any  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ ; here for any  $n \in \mathbb{Z}$ ,

$$o_n^{\mathcal{Y}}(w_{(1)} \otimes w_{(2)}) = (w_{(1)})_{n - h_3 + h_1 + h_2} w_{(2)}$$

for  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ . If  $w_{(1)}$  and  $w_{(2)}$  are homogenous, then

$$\text{wt } o_n^{\mathcal{Y}}(w_{(1)} \otimes w_{(2)}) = \text{wt } w_{(1)} + \text{wt } w_{(2)} + h_3 - h_1 - h_2 - n - 1$$

for any  $n \in \mathbb{Z}$ . In particular, for any  $w_{(1)} \in W_1$ , we have a linear map

$$o_{\mathcal{Y}}(w_{(1)}) \in \text{Hom}((W_2)_{(h_2)}, (W_3)_{(h_3)})$$

defined by

$$o_{\mathcal{Y}}(w_{(1)}) : u_{(2)} \mapsto o_{\text{wt } w_{(1)} - h_1 - 1}^{\mathcal{Y}}(w_{(1)} \otimes u_{(2)})$$

for homogeneous  $w_{(1)} \in W_1$  and  $u_{(2)} \in W_{(h_2)}$ .

From [16], the linear map  $o_{\mathcal{Y}}(w_{(1)}) = 0$  when  $w_{(1)} \in O(W_1)$ , so for any  $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$ , we have an  $A(V)$ -homomorphism

$$\pi(\mathcal{Y}) : A(W_1) \otimes_{A(V)} (W_2)_{(h_2)} \rightarrow (W_3)_{(h_3)}$$

defined by

$$\pi(\mathcal{Y})((w_{(1)} + O(W_1)) \otimes u_{(2)}) = o_{\mathcal{Y}}(w_{(1)}) \cdot u_{(2)}$$

for  $w_{(1)} \in W_1$  and  $u_{(2)} \in (W_2)_{(h_2)}$ . In certain cases, the map  $\mathcal{Y} \mapsto \pi(\mathcal{Y})$  is an isomorphism. In fact, from Theorem 2.1 in [47] (which is a correction and generalization of Theorem 1.5.3 in [16]) we have

**Theorem 4.4.** *Suppose  $W_1 \cong S(M_1)$  and  $W_2 \cong S(M_2)$  where  $M_1$  and  $M_2$  are finite-dimensional irreducible  $A(V)$ -modules; suppose also that  $W_3 \cong S(M_3)'$  where  $M_3$  is a finite-dimensional irreducible  $A(V)$ -module. Then  $\pi$  is a linear isomorphism, so as vector spaces,*

$$\mathcal{V}_{W_1 W_2}^{W_3} \cong \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} (W_2)_{(h_2)}, (W_3)_{(h_3)}).$$

When every  $\mathbb{N}$ -gradable weak  $V$ -module is a direct sum of irreducible  $V$ -modules, both irreducible modules and their (irreducible) contragredients are isomorphic to modules  $S(M)$  where  $M$  is a finite-dimensional irreducible  $A(V)$ -module. Moreover, the lowest weight space of an irreducible module  $W$  is  $T(W)$ . Thus we have:

**Corollary 4.5.** *Assume that every  $\mathbb{N}$ -gradable weak  $V$ -module is a direct sum of irreducible  $V$ -modules and that  $W_1$ ,  $W_2$ , and  $W_3$  are  $V$ -modules. Then as vector spaces,*

$$\mathcal{V}_{W_1 W_2}^{W_3} \cong \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} T(W_2), T(W_3)).$$

We can use Corollary 4.5 to identify the  $P(z)$ -tensor product of  $W_1$  and  $W_2$ . In particular, suppose that the  $A(V)$ -module  $A(W_1) \otimes_{A(V)} T(W_2)$  is finite-dimensional. Then the identity on  $A(W_1) \otimes_{A(V)} T(W_2)$  induces an intertwining operator  $\mathcal{Y}_{\boxtimes}$  of type  $\binom{S(A(W_1) \otimes_{A(V)} T(W_2))}{W_1 \ W_2}$ . Moreover, recalling Proposition 3.6, we have for any  $z \in \mathbb{C}^\times$  a  $P(z)$ -intertwining map  $I_{\mathcal{Y}_{\boxtimes}, 0}$ . Then we have (see also Theorem 6.3.3 in [46]):

**Proposition 4.6.** *In the setting of Corollary 4.5, assume that the  $A(V)$ -module  $A(W_1) \otimes_{A(V)} T(W_2)$  is finite dimensional. Then for any  $z \in \mathbb{C}^\times$ ,  $(S(A(W_1) \otimes_{A(V)} T(W_2)), I_{\mathcal{Y}_{\boxtimes}, 0})$  is a  $P(z)$ -tensor product of  $W_1$  and  $W_2$ .*

*Proof.* Let us use  $M_{1,2}$  to denote the finite-dimensional  $A(V)$ -module  $A(W_1) \otimes_{A(V)} T(W_2)$ . We need to check that  $(S(M_{1,2}), I_{\mathcal{Y}_{\boxtimes}, 0})$  satisfies the universal property of a  $P(z)$ -tensor product. Thus suppose  $W_3$  is a  $V$ -module and  $I$  is an intertwining map of type  $\binom{W_3}{W_1 \ W_2}$ . We need to show that there is a unique  $V$ -module homomorphism  $\eta : S(M_{1,2}) \rightarrow W_3$  such that  $\bar{\eta} \circ I_{\mathcal{Y}_{\boxtimes}, 0} = I$ .

Recalling Proposition 3.6, we have the intertwining operator  $\mathcal{Y}_{I, 0}$  of type  $\binom{W_3}{W_1 \ W_2}$ , and we have the  $A(V)$ -homomorphism

$$\pi(\mathcal{Y}_{I, 0}) : M_{1,2} \rightarrow T(W_3).$$

Suppose also that  $\tau : S(T(W_3)) \rightarrow W_3$  is the unique  $V$ -module isomorphism that equals the identity on  $T(W_3)$ . We set  $\eta = \tau \circ S(\pi(\mathcal{Y}_{I, 0}))$ .

Now,  $\eta \circ \mathcal{Y}_{\boxtimes}$  and  $\mathcal{Y}_{I, 0}$  are two intertwining operators of type  $\binom{W_3}{W_1 \ W_2}$ . Moreover, it is clear from the definitions that

$$\pi(\eta \circ \mathcal{Y}_{\boxtimes}) = \pi(\mathcal{Y}_{I, 0})$$

as  $A(V)$ -module homomorphisms from  $M_{1,2}$  to  $T(W_3)$ . Since  $\pi$  is an isomorphism by Corollary 4.5, it follows that  $\eta \circ \mathcal{Y}_{\boxtimes} = \mathcal{Y}_{I, 0}$ . Then

$$\bar{\eta} \circ I_{\mathcal{Y}_{\boxtimes}, 0} = \bar{\eta} \circ \mathcal{Y}_{\boxtimes}(\cdot, e^{\log z}) \cdot = \mathcal{Y}_{I, 0}(\cdot, e^{\log z}) \cdot = I$$

as well. To show the uniqueness of  $\eta$ , note that if  $\bar{\eta} \circ I_{\mathcal{Y}_{\boxtimes}, 0} = I$ , then Proposition 3.6 implies that  $\eta \circ \mathcal{Y}_{\boxtimes} = \mathcal{Y}_{I, 0}$  as well. Thus  $\pi(\eta \circ \mathcal{Y}_{\boxtimes}) = \pi(\mathcal{Y}_{I, 0})$ , which forces  $T(\eta) = \pi(\mathcal{Y}_{I, 0})$ . Therefore we have

$$\eta = \tau \circ S(T(\eta)) = \tau \circ S(\pi(\mathcal{Y}_{I, 0})),$$

as desired. □

### 4.3 An equivalence of tensor categories

Here we work in the setting of Proposition 4.6, that is, we assume that every  $\mathbb{N}$ -gradable weak  $V$ -module is a direct sum of irreducible  $V$ -modules and that  $A(W_1) \otimes_{A(V)} T(W_2)$  is finite dimensional for any  $V$ -modules  $W_1$  and  $W_2$ . We also assume that  $V - \mathbf{mod}$  equipped with the tensor product  $\boxtimes_{P(1)}$  is a tensor category with associativity isomorphisms  $\mathcal{A}$ , unit object  $V$ , and left and right unit isomorphisms  $l$  and  $r$  (see Section 12.2 in [33] for a detailed description of this tensor category structure). We discuss precisely how the functor  $T$  induces tensor category structure on  $\mathbf{C}^{fin}(A(V))$ , omitting proofs because they are straightforward.

As in Proposition 4.6, for  $V$ -modules  $W_1$  and  $W_2$ , we take

$$W_1 \boxtimes_{P(1)} W_2 = S(A(W_1) \otimes_{A(V)} T(W_2)),$$

and we use  $I_{\boxtimes,0}$  as the tensor product  $P(1)$ -intertwining map. Also, for morphisms  $f_1 : W_1 \rightarrow \widetilde{W}_1$  and  $f_2 : W_2 \rightarrow \widetilde{W}_2$  in  $V - \mathbf{mod}$ , the tensor product morphism is induced by the universal property of the tensor product:  $f_1 \boxtimes_{P(1)} f_2$  is the unique morphism such that

$$\overline{f_1 \boxtimes_{P(1)} f_2}(\mathcal{Y}_{\boxtimes}(\cdot, 1) \cdot) = \widetilde{\mathcal{Y}}_{\boxtimes}(f_1(\cdot), 1) f_2(\cdot),$$

where  $\widetilde{\mathcal{Y}}_{\boxtimes}$  denotes the tensor product intertwining operator of type  $(\begin{smallmatrix} \widetilde{W}_1 \boxtimes_{P(1)} \widetilde{W}_2 \\ \widetilde{W}_1 \quad \widetilde{W}_2 \end{smallmatrix})$ . By the definition of  $\mathcal{Y}_{\boxtimes}$  and  $\widetilde{\mathcal{Y}}_{\boxtimes}$ , this means  $f_1 \boxtimes_{P(1)} f_2$  is the morphism

$$f_1 \boxtimes_{P(1)} f_2 = S(A(f_1) \otimes T(f_2)) : S(A(W_1) \otimes_{A(V)} T(W_2)) \rightarrow S(A(\widetilde{W}_1) \otimes_{A(V)} T(\widetilde{W}_2)),$$

where  $A(f_1) : A(W_1) \rightarrow A(\widetilde{W}_1)$  is the  $A(V)$ -bimodule homomorphism induced by  $f_1$ .

Now we can define a tensor product bifunctor  $\boxtimes$  on  $\mathbf{C}^{fin}(A(V))$  as follows. For finite-dimensional  $A(V)$ -modules  $U_1$  and  $U_2$ , define

$$U_1 \boxtimes U_2 = A(S(U_1)) \otimes_{A(V)} U_2,$$

a finite-dimensional  $A(V)$ -module by our assumptions. Also, for morphisms  $f_1 : U_1 \rightarrow \widetilde{U}_1$  and  $f_2 : U_2 \rightarrow \widetilde{U}_2$  in  $\mathbf{C}^{fin}(A(V))$ , we define

$$f_1 \boxtimes f_2 = A(S(f_1)) \otimes f_2 : A(S(U_1)) \otimes_{A(V)} U_2 \rightarrow A(S(\widetilde{U}_1)) \otimes_{A(V)} \widetilde{U}_2. \quad (4.1)$$

Note that with these definitions,

$$S(U_1 \boxtimes U_2) = S(U_1) \boxtimes_{P(1)} S(U_2), \quad (4.2)$$

so because  $T \circ S = 1_{\mathbf{C}^{fin}(A(V))}$ ,

$$U_1 \boxtimes U_2 = T(S(U_1) \boxtimes_{P(1)} S(U_2)) \quad (4.3)$$

for any finite-dimensional  $A(V)$ -modules  $U_1$  and  $U_2$ .

The relations (4.2) and (4.3) imply that for objects  $U_1$ ,  $U_2$ , and  $U_3$  in  $\mathbf{C}^{fin}(A(V))$ ,

$$U_1 \boxtimes (U_2 \boxtimes U_3) = T(S(U_1) \boxtimes_{P(1)} (S(U_2) \boxtimes_{P(1)} S(U_3)))$$

and

$$(U_1 \boxtimes U_2) \boxtimes U_3 = T((S(U_1) \boxtimes_{P(1)} S(U_2)) \boxtimes_{P(1)} S(U_3)).$$

Thus we can take the associativity isomorphism for  $U_1$ ,  $U_2$ , and  $U_3$  to be

$$\mathcal{A}_{U_1, U_2, U_3} = T(\mathcal{A}_{S(U_1), S(U_2), S(U_3)}).$$

That these associativity isomorphisms give a natural isomorphism from  $\boxtimes \circ (1_{\mathbf{C}^{fin}(A(V))} \times \boxtimes)$  to  $\boxtimes \circ (\boxtimes \times 1_{\mathbf{C}^{fin}(A(V))})$  and satisfy the pentagon axiom follows easily from these properties for the associativity isomorphisms in  $V - \mathbf{mod}$ .

Now, as in the proof of Proposition 4.6, let  $\tau_W$  for a  $V$ -module  $W$  denote the unique isomorphism from  $S(T(W))$  to  $W$  that equals the identity on the top level  $T(W)$ . Notice that for  $V$ -modules  $W_1$  and  $W_2$ ,

$$T(W_1) \boxtimes T(W_2) = A(S(T(W_1))) \otimes_{A(V)} T(W_2)$$

while

$$T(W_1 \boxtimes_{P(1)} W_2) = A(W_1) \otimes_{A(V)} T(W_2).$$

Then we can define

$$M_{W_1, W_2} : T(W_1) \boxtimes T(W_2) \rightarrow T(W_1 \boxtimes_{P(1)} W_2)$$

to be  $M_{W_1, W_2} = A(\tau_{W_1}) \otimes 1_{T(W_2)}$ . Thus we obtain a natural isomorphism  $M$  from  $\boxtimes \circ (T \times T)$  to  $T \circ \boxtimes_{P(1)}$ .

Next, we take the unit object of  $\mathbf{C}^{fin}(A(V))$  to be  $T(V)$ . The natural isomorphism  $M$  then allows us to define unit isomorphisms in  $\mathbf{C}^{fin}(A(V))$ . In particular, for a finite-dimensional  $A(V)$ -module  $U$ , we define the left unit isomorphism  $l_U$  to be the composition

$$T(V) \boxtimes U = T(V) \boxtimes T(S(U)) \xrightarrow{M_{V, S(U)}} T(V \boxtimes_{P(1)} S(U)) \xrightarrow{T(l_{S(U)})} T(S(U)) = U \quad (4.4)$$

and the right unit isomorphism  $r_U$  to be the composition

$$U \boxtimes T(V) = T(S(U)) \boxtimes T(V) \xrightarrow{M_{S(U), V}} T(S(U) \boxtimes_{P(1)} V) \xrightarrow{T(r_{S(U)})} T(S(U)) = U. \quad (4.5)$$

We recall the notion of equivalence of tensor categories from, for example, [2] or [37]. Then it is straightforward to prove:

**Theorem 4.7.** *Under the assumptions of this subsection, the category  $\mathbf{C}^{fin}(A(V))$  equipped with the tensor product  $\boxtimes$ , unit object  $T(V)$ , and associativity and unit isomorphisms as described above is a tensor category. Moreover,  $(T, M, 1_{T(V)})$  defines a tensor equivalence from  $V - \mathbf{mod}$  to  $\mathbf{C}^{fin}(A(V))$ .*

#### 4.4 Application to $L_{\hat{\mathfrak{g}}}(\ell, 0)$

We now apply the results of this section to the vertex operator algebra  $L_{\hat{\mathfrak{g}}}(\ell, 0)$  when  $\ell \in \mathbb{N}$ . It was shown in Theorem 3.1.3 of [16] that every  $\mathbb{N}$ -gradable weak  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -module is a direct

sum of irreducible  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules; in particular,  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  is semisimple. First, from [16],

$$A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_{\theta}^{\ell+1} \rangle \quad (4.6)$$

as an associative algebra, where  $x_{\theta}$  is a root vector for the longest root  $\theta$  of  $\mathfrak{g}$ ; the isomorphism is determined by

$$g + \langle x_{\theta}^{\ell+1} \rangle \mapsto g(-1)\mathbf{1} + O(L_{\widehat{\mathfrak{g}}}(\ell, 0))$$

for  $g \in \mathfrak{g}$ . Thus if we set  $\mathbf{D}(\mathfrak{g}, \ell)$  to be the category of finite-dimensional  $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ -modules, it is clear that  $\mathbf{D}(\mathfrak{g}, \ell)$  is simply the full subcategory of finite-dimensional  $\mathfrak{g}$ -modules on which  $x_{\theta}^{\ell+1}$  acts trivially. By Theorem 4.3, the functor

$$T : L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod} \rightarrow \mathbf{D}(\mathfrak{g}, \ell)$$

is an equivalence of categories. Note that if  $U$  is an object of  $\mathbf{D}(\mathfrak{g}, \ell)$ , then  $T(L_{\widehat{\mathfrak{g}}}(\ell, U)) = U$  and  $S(U) \cong L_{\widehat{\mathfrak{g}}}(\ell, U)$ .

Next from [16], for an irreducible  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module  $L_{\widehat{\mathfrak{g}}}(\ell, \lambda)$ , the corresponding  $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ -bimodule is

$$A(L_{\widehat{\mathfrak{g}}}(\ell, \lambda)) \cong (L_{\lambda} \otimes U(\mathfrak{g})) / \langle v_{\lambda} \otimes x_{\theta}^{\ell - \langle \lambda, \theta \rangle + 1} \rangle, \quad (4.7)$$

where  $v_{\lambda}$  is a highest weight vector of  $L_{\lambda}$  and  $\langle \cdot \rangle$  indicates the sub-bimodule generated by an element; the isomorphism is determined by

$$u \otimes g_1 \cdots g_n + \langle v_{\lambda} \otimes x_{\theta}^{\ell - \langle \lambda, \theta \rangle + 1} \rangle \mapsto g_n(-1) \cdots g_1(-1)u + O(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda})) \quad (4.8)$$

for  $u \in L_{\lambda}$  and  $g_1, \dots, g_n \in \mathfrak{g}$ . The  $A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_{\theta}^{\ell+1} \rangle$ -bimodule structure on  $A(L_{\widehat{\mathfrak{g}}}(\ell, \lambda))$  is induced by the following  $U(\mathfrak{g})$ -bimodule structure on  $L_{\lambda} \otimes U(\mathfrak{g})$ :

$$x \cdot (v \otimes y) = (x \cdot v) \otimes y + v \otimes xy \quad (4.9)$$

for  $x \in \mathfrak{g}$ ,  $y \in U(\mathfrak{g})$ ,  $v \in L_{\lambda}$ , and

$$(v \otimes y) \cdot x = v \otimes yx. \quad (4.10)$$

We can now identify the tensor product in  $\mathbf{D}(\mathfrak{g}, \ell)$  with a quotient of the usual tensor product. For objects  $U_1$  and  $U_2$  of  $\mathbf{D}(\mathfrak{g}, \ell)$ , let  $W_{U_1, U_2}^{(\ell)}$  denote the  $\mathfrak{g}$ -submodule of  $U_1 \otimes U_2$  generated by vectors of the form  $v_{\lambda} \otimes x_{\theta}^{\ell - \langle \lambda, \theta \rangle + 1} \cdot w$  where  $v_{\lambda}$  is any highest weight vector (of weight  $\lambda$ ) in  $U_1$  and  $w$  is any vector in  $U_2$ .

**Proposition 4.8.** *For objects  $U_1$  and  $U_2$  in  $\mathbf{D}(\mathfrak{g}, \ell)$ , there is an  $A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_{\theta}^{\ell+1} \rangle$  isomorphism*

$$\begin{aligned} \Phi_{U_1, U_2} : (U_1 \otimes U_2) / W_{U_1, U_2}^{(\ell)} &\rightarrow A(S(U_1)) \otimes_{A(L_{\widehat{\mathfrak{g}}}(\ell, 0))} U_2 \\ u_{(1)} \otimes u_{(2)} + W_{U_1, U_2}^{(\ell)} &\mapsto (u_{(1)} + O(S(U_1))) \otimes u_{(2)} \end{aligned} \quad (4.11)$$

*Proof.* Since finite-dimensional  $\mathfrak{g}$ -modules are completely reducible, we may take  $U_1 = L_{\lambda_1}$  where  $\langle \lambda_1, \theta \rangle \leq \ell$ . Then using the identifications (4.7), (4.8), (4.9), and (4.10), we have a  $\mathfrak{g}$ -module homomorphism

$$\begin{aligned} \widetilde{\Phi}_{U_1, U_2} : U_1 \otimes U_2 &\rightarrow (U_1 \otimes U(\mathfrak{g})) / \langle v_{\lambda} \otimes x_{\theta}^{\ell - \langle \lambda_1, \theta \rangle + 1} \rangle \otimes U_2 \\ u_{(1)} \otimes u_{(2)} &\mapsto (u_{(1)} \otimes 1 + \langle v_{\lambda} \otimes x_{\theta}^{\ell - \langle \lambda_1, \theta \rangle + 1} \rangle) \otimes u_{(2)} \end{aligned}$$

which contains  $W_{U_1, U_2}^{(\ell)}$  in its kernel, so we have the desired homomorphism  $\Phi_{U_1, U_2}$ . It is straightforward to show that  $\Phi_{U_1, U_2}$  has an inverse, so it is an isomorphism. See Lemma 4.1 in [48] for more details; see also Theorem 3.2.3 in [16] and the related Theorem 6.5 in [11].  $\square$

Due to the preceding proposition, we can take

$$U_1 \boxtimes U_2 = (U_1 \otimes U_2) / W_{U_1, U_2}^{(\ell)} \quad (4.12)$$

for  $\mathfrak{g}$ -modules  $U_1$  and  $U_2$  in  $\mathbf{D}(\mathfrak{g}, \ell)$ , recalling the  $\boxtimes$  notation of the previous subsection. Note that if  $f_1 : U_1 \rightarrow \tilde{U}_1$  and  $f_2 : U_2 \rightarrow \tilde{U}_2$  are  $\mathfrak{g}$ -module homomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$ ,  $f_1 \otimes f_2$  takes generators of  $W_{U_1, U_2}^{(\ell)}$  to generators of  $W_{\tilde{U}_1, \tilde{U}_2}^{(\ell)}$ , so  $f_1 \otimes f_2$  induces a  $\mathfrak{g}$ -module homomorphism

$$f_1 \boxtimes f_2 : U_1 \boxtimes U_2 \rightarrow \tilde{U}_1 \boxtimes \tilde{U}_2.$$

Then (4.1) and (4.11) show that the isomorphisms  $\Phi_{U_1, U_2}$  define a natural isomorphism.

**Remark 4.9.** Our realization (4.12) of  $U_1 \boxtimes U_2$  is not especially useful for calculating fusion rules, that is, the multiplicities of irreducible  $\mathfrak{g}$ -modules in  $U_1 \boxtimes U_2$ , since the submodule  $W_{U_1, U_2}^{(\ell)}$  is usually difficult to calculate explicitly (but note that Theorem 6.2 in [11] is an interesting formula for fusion rules likewise derived using results in [16]). The main reason we use (4.12) is that it allows us to realize the triple tensor products  $U_1 \boxtimes (U_2 \boxtimes U_3)$  and  $(U_1 \boxtimes U_2) \boxtimes U_3$ , for objects  $U_1$ ,  $U_2$ , and  $U_3$  of  $\mathbf{D}(\mathfrak{g}, \ell)$ , as quotients of  $U_1 \otimes U_2 \otimes U_3$ . In the following sections, we will use this to realize the associativity isomorphism  $\mathcal{A}_{U_1, U_2, U_3}$  as the isomorphism induced from a certain automorphism of  $U_1 \otimes U_2 \otimes U_3$ .

Now for  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules  $W_1$  and  $W_2$ , we recall from Proposition 4.6 the intertwining operator  $\mathcal{Y}_{\boxtimes}$  of type  $\left( \begin{smallmatrix} S(A(W_1) \otimes_{A(L_{\hat{\mathfrak{g}}}(\ell, 0))} T(W_2)) \\ W_1 \quad W_2 \end{smallmatrix} \right)$  and obtain:

**Proposition 4.10.** *If  $W_1$  and  $W_2$  are  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules and  $z \in \mathbb{C}^\times$ ,  $S(A(W_1) \otimes_{A(L_{\hat{\mathfrak{g}}}(\ell, 0))} T(W_2))$  equipped with the  $P(z)$ -intertwining map  $I_{\mathcal{Y}_{\boxtimes}, 0}$  is a  $P(z)$ -tensor product of  $W_1$  and  $W_2$ .*

Thus in the tensor category of  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules given by [22]-[25], [17], we may take the tensor product  $\boxtimes_{P(1)}$  of  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules  $W_1$  and  $W_2$  to be

$$W_1 \boxtimes_{P(1)} W_2 = S(A(W_1) \otimes_{A(L_{\hat{\mathfrak{g}}}(\ell, 0))} T(W_2)) \cong L_{\hat{\mathfrak{g}}}(\ell, T(W_1) \boxtimes T(W_2))$$

equipped with the  $P(1)$ -intertwining map

$$I_{\mathcal{Y}_{\boxtimes}, 0} = \mathcal{Y}_{\boxtimes}(\cdot, 1) \cdot.$$

Moreover, by Theorem 4.7 and the natural isomorphism of Proposition 4.8, the equivalence of categories  $T$  is in fact an equivalence of tensor categories, where  $\mathbf{D}(\mathfrak{g}, \ell)$  is equipped with the tensor product  $\boxtimes$  of (4.12).

The unit object of  $\mathbf{D}(\mathfrak{g}, \ell)$  is  $T(L_{\hat{\mathfrak{g}}}(\ell, 0)) = \mathbb{C}\mathbf{1}$ , the trivial one-dimensional  $\mathfrak{g}$ -module. It is easy to see from (4.4), (4.5), (4.11), and the definition of the unit isomorphisms in

$L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  (see for example Section 12.2 in [33]), that the unit isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$  are the obvious ones

$$\begin{aligned} l_U : \mathbb{C}\mathbf{1} \boxtimes U &= \mathbb{C}\mathbf{1} \otimes U \rightarrow U \\ \mathbf{1} \otimes u &\mapsto u \end{aligned}$$

and

$$\begin{aligned} r_U : U \boxtimes \mathbb{C}\mathbf{1} &= U \otimes \mathbb{C}\mathbf{1} \rightarrow U \\ u \otimes \mathbf{1} &\mapsto u \end{aligned}$$

for  $\mathfrak{g}$ -modules  $U$  in  $\mathbf{D}(\mathfrak{g}, \ell)$ . (Note that both  $W_{\mathbb{C}\mathbf{1}, U}^{(\ell)}$  and  $W_{U, \mathbb{C}\mathbf{1}}^{(\ell)}$  are zero.) The remainder of this paper is devoted to a description of the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$ .

## 5 The Knizhnik-Zamolodchikov equations

The goal of this section and the next is to identify the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$  with certain isomorphisms of  $\mathfrak{g}$ -modules obtained from solutions to Knizhnik-Zamolodchikov (KZ) equations ([43]; see also [25]).

### 5.1 Formal KZ equations

We start by deriving a system of (formal) differential equations satisfied by an iterate of intertwining operators. Such equations were first derived in [43]. We will also need a similar system of equations for a product of intertwining operators; we will not derive them here, since a vertex algebraic derivation may be found in [25].

First we consider the action of  $L(-1)$  on a  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module  $W$ . If  $w \in W$  satisfies  $g(n)w = 0$  for any  $g \in \mathfrak{g}$  and  $n > 0$ , then (2.5) implies

$$L(-1)w = \frac{1}{\ell + h^\vee} \sum_{i=1}^{\dim \mathfrak{g}} \gamma_i(-1) \gamma_i(0)w. \quad (5.1)$$

We shall use the  $L(-1)$ -derivative property (3.3) for intertwining operators and (5.1) to derive the KZ equations for an iterate of intertwining operators.

We will need the commutator formula (3.4) for intertwining operators. Suppose  $W_1, W_2$ , and  $W_3$  are  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -modules and  $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$ . If we take  $v = g(-1)\mathbf{1}$  for  $g \in \mathfrak{g}$  in (3.4), we obtain

$$\begin{aligned} [g(x_1), \mathcal{Y}(w_{(1)}, x_2)] &= \text{Res}_{x_0} x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(g(x_0)w_{(1)}, x_2) \\ &= \sum_{i \geq 0} \frac{(-1)^i}{i!} \left( \frac{\partial}{\partial x_1} \right)^i \left( x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) \right) \mathcal{Y}(g(i)w_{(1)}, x_2) \end{aligned} \quad (5.2)$$

for  $w_{(1)} \in W_1$ . If we further extract the coefficient of  $x_1^{-n-1}$  in (5.2), we obtain

$$[g(n), \mathcal{Y}(w_{(1)}, x_2)] = \sum_{i \geq 0} \binom{n}{i} x_2^{n-i} \mathcal{Y}(g(i)w_{(1)}, x_2) \quad (5.3)$$

for  $w_{(1)} \in W_1$ . When  $w_{(1)} \in W_1$  satisfies  $g(i)w = 0$  for  $i > 0$ , (5.3) simplifies to

$$[g(n), \mathcal{Y}(w_{(1)}, x)] = x^n \mathcal{Y}(g(0)w_{(1)}, x) \quad (5.4)$$

for any  $g \in \mathfrak{g}$  and  $n \in \mathbb{Z}$ . We shall also need the iterate formula for intertwining operators. When we take  $v = g(-1)\mathbf{1}$  for  $g \in \mathfrak{g}$  in (3.5), we obtain

$$\begin{aligned} \mathcal{Y}(g(n)w_{(1)}, x_2) &= \text{Res}_{x_1} ((x_1 - x_2)^n g(x_1) \mathcal{Y}(w_{(1)}, x_2) - (-x_2 + x_1)^n \mathcal{Y}(w_{(1)}, x_2) g(x_1)) \\ &= \sum_{i \geq 0} (-1)^i \binom{n}{i} (x_2^i g(n-i) \mathcal{Y}(w_{(1)}, x_2) - (-1)^n x_2^{n-i} \mathcal{Y}(w_{(1)}, x_2) g(i)) \end{aligned} \quad (5.5)$$

for  $w_{(1)} \in W_1$ .

Now suppose  $W_1, W_2, W_3, W_4$  and  $M_2$  are  $L_{\mathfrak{g}}(\ell, 0)$ -modules,  $\mathcal{Y}^1$  is an intertwining operator of type  $\binom{W_4}{M_2 W_3}$ , and  $\mathcal{Y}^2$  is an intertwining operator of type  $\binom{M_2}{W_1 W_2}$ . We use (5.5) to obtain:

**Lemma 5.1.** *For  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ ,  $u_{(3)} \in T(W_3)$ ,  $u'_{(4)} \in T(W'_4)$ , and  $g \in \mathfrak{g}$ ,*

$$\begin{aligned} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(g(-1)u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)} \rangle &= x_0^{-1} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)g(0)u_{(2)}, x_2)u_{(3)} \rangle \\ &\quad + (x_2 + x_0)^{-1} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)g(0)u_{(3)} \rangle \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)g(-1)u_{(2)}, x_2)u_{(3)} \rangle &= -x_0^{-1} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(g(0)u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)} \rangle \\ &\quad + x_2^{-1} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)g(0)u_{(3)} \rangle. \end{aligned} \quad (5.7)$$

*Proof.* To prove (5.6), we first note that (5.5) and the fact that  $g(i)u_{(2)} = 0$  for  $i > 0$  implies that

$$\mathcal{Y}^2(g(-1)u_{(1)}, x_0)u_{(2)} = x_0^{-1} \mathcal{Y}^2(u_{(1)}, x_0)g(0)u_{(2)} + \sum_{i \geq 0} x_0^i g(-i-1) \mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}. \quad (5.8)$$

Then applying (5.5) again as well as the fact that  $g(j)u_{(3)} = 0$  for  $j > 0$ ,

$$\begin{aligned} \mathcal{Y}^1(g(-i-1) \mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)} &= (-1)^i x_2^{-i-1} \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)g(0)u_{(3)} \\ &\quad + \sum_{j \geq 0} (-1)^j \binom{-i-1}{j} x_2^j g(-i-j-1) \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)}. \end{aligned} \quad (5.9)$$

Putting (5.8) and (5.9) together,

$$\begin{aligned} \mathcal{Y}^1(\mathcal{Y}^2(g(-1)u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)} &= x_0^{-1} \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)g(0)u_{(2)}, x_2)u_{(3)} \\ &\quad + \sum_{i \geq 0} (-1)^i x_2^{-1-i} x_0^i \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)g(0)u_{(3)} \\ &\quad + \sum_{i \geq 0} \sum_{j \geq 0} (-1)^j \binom{-i-1}{j} x_2^j g(-i-j-1) \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)}. \end{aligned} \quad (5.10)$$

Since the adjoint operator of  $g(-i-j-1)$  is  $-g(i+j+1)$ , the third term on the right side of (5.10) disappears when (5.10) is paired with  $u'_{(4)} \in T(W'_4)$ . Since also

$$\sum_{i \geq 0} (-1)^i x_2^{-i-1} x_0^i = (x_2 + x_0)^{-1},$$

(5.6) follows.

To prove (5.7), we first use (5.4) to obtain

$$\mathcal{Y}^2(u_{(1)}, x_0)g(-1)u_{(2)} = g(-1)\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)} - x_0^{-1}\mathcal{Y}^2(g(0)u_{(1)}, x_0)u_{(2)}. \quad (5.11)$$

Then by (5.5) and the fact that  $g(i)u_{(3)} = 0$  for  $i > 0$ ,

$$\begin{aligned} \mathcal{Y}^1(g(-1)\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)} &= x_2^{-1}\mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)g(0)u_{(3)} \\ &+ \sum_{i \geq 0} x_2^i g(-i-1)\mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)}. \end{aligned} \quad (5.12)$$

Putting (5.11) and (5.12) together,

$$\begin{aligned} \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)g(-1)u_{(2)}, x_2)u_{(3)} &= -x_0^{-1}\mathcal{Y}^1(\mathcal{Y}^2(g(0)u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)} \\ &+ x_2^{-1}\mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)g(0)u_{(3)} + \sum_{i \geq 0} x_2^i g(-i-1)\mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_2)u_{(3)}. \end{aligned} \quad (5.13)$$

Since the adjoint of the operator  $g(-i-1)$  is  $-g(i+1)$ , the third term on the right side of (5.13) disappears when (5.13) is paired with  $u'_{(4)} \in T(W'_4)$ , and (5.7) follows.  $\square$

We now define an operator  $\Omega_{1,2}$  on  $(T(W_1) \otimes T(W_2) \otimes T(W_3))^*$  by

$$(\Omega_{1,2}F)(u_{(1)} \otimes u_{(2)} \otimes u_{(3)}) = \sum_{i=1}^{\dim \mathfrak{g}} F(\gamma_i(0)u_{(1)} \otimes \gamma_i(0)u_{(2)} \otimes u_{(3)}) \quad (5.14)$$

for any  $F \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^*$ , recalling that  $\{\gamma_i\}$  is an orthonormal basis for  $\mathfrak{g}$ . We analogously define operators  $\Omega_{1,3}$  and  $\Omega_{2,3}$  on  $(T(W_1) \otimes T(W_2) \otimes T(W_3))^*$  in the obvious way. The operators  $\Omega_{1,2}$ ,  $\Omega_{1,3}$ , and  $\Omega_{2,3}$  also have natural extensions to operators on  $(T(W_1) \otimes T(W_2) \otimes T(W_3))^*\{x_0, x_2\}$ . We can now derive the KZ equations:

**Theorem 5.2.** *For any  $u'_{(4)} \in T(W'_4)$ , define*

$$\varphi_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) = \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(\cdot, x_0)\cdot, x_2)\cdot \rangle \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^*\{x_0, x_2\}.$$

*Then  $\varphi_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)})$  satisfies the system of formal differential equations*

$$(\ell + h^\vee) \frac{\partial \varphi}{\partial x_0} = \left( \frac{\Omega_{1,2}}{x_0} + \frac{\Omega_{1,3}}{x_2 + x_0} \right) \varphi \quad (5.15)$$

$$(\ell + h^\vee) \frac{\partial \varphi}{\partial x_2} = \left( \frac{\Omega_{2,3}}{x_2} + \frac{\Omega_{1,3}}{x_2 + x_0} \right) \varphi. \quad (5.16)$$

Similarly, suppose  $M_1$  is an  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module,  $\mathcal{Y}_1$  is an intertwining operator of type  $\binom{W_4}{W_1 M_1}$  and  $\mathcal{Y}_2$  is an intertwining operator of type  $\binom{M_1}{W_1 W_2}$ ; for any  $u'_{(4)} \in T(W'_4)$ , set

$$\psi_{\mathcal{Y}_1, \mathcal{Y}_2}(u'_{(4)}) = \langle u'_{(4)}, \mathcal{Y}_1(\cdot, x_1) \mathcal{Y}_2(\cdot, x_2) \cdot \rangle \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^* \{x_1, x_2\}.$$

Then  $\psi_{\mathcal{Y}_1, \mathcal{Y}_2}(u'_{(4)})$  satisfies the system of formal differential equations

$$(\ell + h^\vee) \frac{\partial \psi}{\partial x_1} = \left( \frac{\Omega_{1,2}}{x_1 - x_2} + \frac{\Omega_{1,3}}{x_1} \right) \psi \quad (5.17)$$

$$(\ell + h^\vee) \frac{\partial \psi}{\partial x_2} = \left( \frac{\Omega_{2,3}}{x_2} - \frac{\Omega_{1,2}}{x_1 - x_2} \right) \psi. \quad (5.18)$$

*Proof.* To derive (5.15), we use the  $L(-1)$ -derivative property (3.3), (5.1), and (5.6) to obtain

$$\begin{aligned} (\ell + h^\vee) \frac{\partial}{\partial x_0} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0) u_{(2)}, x_2) u_{(3)} \rangle \\ &= (\ell + h^\vee) \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(L(-1)u_{(1)}, x_0) u_{(2)}, x_2) u_{(3)} \rangle \\ &= \sum_{i=1}^{\dim \mathfrak{g}} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(\gamma_i(-1) \gamma_i(0) u_{(1)}, x_0) u_{(2)}, x_2) u_{(3)} \rangle \\ &= \sum_{i=1}^{\dim \mathfrak{g}} x_0^{-1} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(\gamma_i(0) u_{(1)}, x_0) \gamma_i(0) u_{(2)}, x_2) u_{(3)} \rangle \\ &\quad + \sum_{i=1}^{\dim \mathfrak{g}} (x_2 + x_0)^{-1} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(\gamma_i(0) u_{(1)}, x_0) u_{(2)}, x_2) \gamma_i(0) u_{(3)} \rangle \end{aligned} \quad (5.19)$$

for any  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ ,  $u_{(3)} \in T(W_3)$ , and  $u'_{(4)} \in T(W'_4)$ , and (5.15) follows.

To derive (5.16), we will need the  $L(-1)$ -commutator formula

$$[L(-1), \mathcal{Y}(w, x)] = \mathcal{Y}(L(-1)w, x)$$

which follows from the Jacobi identity (3.2) with  $v = \omega$  and holds for any intertwining operator  $\mathcal{Y}$ . Using this, the  $L(-1)$ -derivative property (3.3), and (5.1),

$$\begin{aligned} (\ell + h^\vee) \frac{\partial}{\partial x_2} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0) u_{(2)}, x_2) u_{(3)} \rangle \\ &= (\ell + h^\vee) \langle u'_{(4)}, \mathcal{Y}^1(L(-1) \mathcal{Y}^2(u_{(1)}, x_0) u_{(2)}, x_2) u_{(3)} \rangle \\ &= (\ell + h^\vee) \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0) L(-1) u_{(2)}, x_2) u_{(3)} \rangle \\ &\quad + (\ell + h^\vee) \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(L(-1) u_{(1)}, x_0) u_{(2)}, x_2) u_{(3)} \rangle \\ &= \sum_{i=1}^{\dim \mathfrak{g}} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0) \gamma_i(-1) \gamma_i(0) u_{(2)}, x_2) u_{(3)} \rangle \\ &\quad + (\ell + h^\vee) \frac{\partial}{\partial x_0} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0) u_{(2)}, x_2) u_{(3)} \rangle \end{aligned} \quad (5.20)$$

for any  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ ,  $u_{(3)} \in T(W_3)$ , and  $u'_{(4)} \in T(W'_4)$ . Using (5.7), the first term on the right side of (5.20) becomes

$$\begin{aligned} & x_2^{-1} \sum_{i=1}^{\dim \mathfrak{g}} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0) \gamma_i(0) u_{(2)}, x_2) \gamma_i(0) u_{(3)} \rangle \\ & - x_0^{-1} \sum_{i=1}^{\dim \mathfrak{g}} \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(\gamma_i(0) u_{(1)}, x_0) \gamma_i(0) u_{(2)}, x_2) u_{(3)} \rangle. \end{aligned}$$

Combining this with (5.19) and (5.20) yields (5.16).

The derivation of (5.17) and (5.18), which requires an analogue of Lemma 5.1, is analogous, and we omit it here. See for instance [25] for details.  $\square$

**Remark 5.3.** The equation (5.15) is heuristically the same as (5.17) under the identification  $x_0 = x_1 - x_2$ . But one cannot substitute  $x_0 = x_1 - x_2$  in (5.15) because  $(x_2 + (x_1 - x_2))^{-1}$  is not a well-defined formal series.

Although we cannot substitute  $x_1 - x_2$  for  $x_0$  in  $\varphi_{\mathcal{Y}^1, \mathcal{Y}^2} = \langle \cdot, \mathcal{Y}^1(\mathcal{Y}^2(\cdot, x_0) \cdot, x_2) \cdot \rangle$  or in (5.15)-(5.16), we can substitute  $x_2 = x_1 - x_0$ . Then we have

**Corollary 5.4.** *For any  $u'_{(4)} \in T(W'_4)$ , define*

$$\tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) = \varphi_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)})|_{x_2=x_1-x_0} = \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(\cdot, x_0) \cdot, x_1 - x_0) \cdot \rangle.$$

*Then  $\tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)})$  satisfies the system of formal differential equations*

$$(\ell + h^\vee) \frac{\partial \varphi}{\partial x_0} = \left( \frac{\Omega_{1,2}}{x_0} - \frac{\Omega_{2,3}}{x_1 - x_0} \right) \varphi \quad (5.21)$$

$$(\ell + h^\vee) \frac{\partial \varphi}{\partial x_1} = \left( \frac{\Omega_{2,3}}{x_1 - x_0} + \frac{\Omega_{1,3}}{x_1} \right) \varphi. \quad (5.22)$$

*Proof.* This follows from (5.15) and (5.16) since

$$\frac{\partial}{\partial x_0} \tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) = \left( \frac{\partial}{\partial x_0} \varphi_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) \right) \Big|_{x_2=x_1-x_0} - \left( \frac{\partial}{\partial x_2} \varphi_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) \right) \Big|_{x_2=x_1-x_0}$$

and

$$\frac{\partial}{\partial x_1} \tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) = \left( \frac{\partial}{\partial x_2} \varphi_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) \right) \Big|_{x_2=x_1-x_0}$$

for any  $u'_{(4)} \in T(W'_4)$ .  $\square$

## 5.2 Reduction to one variable

Suppose  $W_1$ ,  $W_2$ , and  $W_3$  are irreducible  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules. Then for each  $i = 1, 2, 3$ , there is some  $h_i \in \mathbb{Q}$  which is the lowest conformal weight of  $W_i$ , so that the conformal weights of  $W_i$  are contained in  $h_i + \mathbb{N}$ . In fact, we have  $h_i = h_{\lambda_i, \ell}$  (recall (2.6)) for some dominant integral weight  $\lambda_i$  of  $\mathfrak{g}$ .

Now suppose  $\mathcal{Y}$  is an intertwining operator of type  $\binom{W_3}{W_1 W_2}$ . By Remark 5.4.4 in [14], we can write

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{Z}} o_n^{\mathcal{Y}}(w_{(1)} \otimes w_{(2)})x^{h-n-1}$$

where  $h = h_3 - h_1 - h_2$ . For each  $n \in \mathbb{Z}$ , we have

$$o_n^{\mathcal{Y}} : (W_1)_{(h_1+k)} \otimes (W_2)_{(h_2+m)} \rightarrow (W_3)_{h_3+k+m-n-1} \quad (5.23)$$

for  $k, m \in \mathbb{N}$ . In particular,  $o_{-1}^{\mathcal{Y}}$  maps  $T(W_1) \otimes T(W_2)$  to  $T(W_3)$ .

Since  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules are completely reducible, an intertwining operator among  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules is a sum of intertwining operators among irreducible  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules. Thus for any intertwining operator of type  $\binom{W_3}{W_1 W_2}$  where  $W_1, W_2$ , and  $W_3$  are not necessarily irreducible, we can define operators  $o_n^{\mathcal{Y}}$  for  $n \in \mathbb{Z}$ . In particular,  $o_{-1}^{\mathcal{Y}}$  maps  $T(W_1) \otimes T(W_2)$  into  $T(W_3)$ .

Now suppose that  $W_1, W_2, W_3, W_4, M_1$ , and  $M_2$  are  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules,  $\mathcal{Y}_1 \in \mathcal{V}_{W_1 M_1}^{W_4}$ ,  $\mathcal{Y}_2 \in \mathcal{V}_{W_2 W_3}^{M_1}$ ,  $\mathcal{Y}^1 \in \mathcal{V}_{M_2 W_3}^{W_4}$ , and  $\mathcal{Y}^2 \in \mathcal{V}_{W_1 W_2}^{M_2}$ . Recalling the operators  $\Omega_{1,2}$ ,  $\Omega_{1,3}$ , and  $\Omega_{2,3}$  on  $(T(W_1) \otimes T(W_2) \otimes T(W_3))^*$  from the previous subsection, we define

$$H = \Omega_{1,2} + \Omega_{1,3} + \Omega_{2,3}$$

and  $\tilde{H} = (\ell + h)^{-1}H$ . We also recall the functionals  $\psi_{\mathcal{Y}_1, \mathcal{Y}_2}(u'_{(4)})$  and  $\tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)})$  for  $u'_{(4)} \in T(W'_4)$  from the previous subsection.

**Proposition 5.5.** *For any  $u'_{(4)} \in T(W'_4)$ , there exist series*

$$F(x), G(x) \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^* \{x\}$$

such that

$$\psi_{\mathcal{Y}_1, \mathcal{Y}_2}(u'_{(4)}) = x_1^{\tilde{H}} F\left(\frac{x_2}{x_1}\right) \quad \text{and} \quad \tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) = x_1^{\tilde{H}} G\left(\frac{x_0}{x_1}\right).$$

*Proof.* Since  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules are completely reducible, we may assume for simplicity that  $W_1, W_2, W_3, W_4, M_1$ , and  $M_2$  are all irreducible, with lowest conformal weights  $h_1, \dots, h_6$ , respectively. We prove the assertion for  $\tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}$ , the proof of the assertion for  $\psi_{\mathcal{Y}_1, \mathcal{Y}_2}$  being similar. For  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ , and  $u_{(3)} \in T(W_3)$ , we have

$$\begin{aligned} \langle \tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle &= \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, x_0)u_{(2)}, x_1 - x_0)u_{(3)} \rangle \\ &= \sum_{m, n \in \mathbb{Z}} \langle u'_{(4)}, o_m^{\mathcal{Y}^1}(o_n^{\mathcal{Y}^2}(u_{(1)} \otimes u_{(2)}) \otimes u_{(3)})x_0^{h_6-h_1-h_2-n-1}(x_1 - x_0)^{h_4-h_6-h_3-m-1} \rangle \\ &= \sum_{n \geq 0} o_{n-1}^{\mathcal{Y}^1}(o_{-n-1}^{\mathcal{Y}^2}(u_{(1)} \otimes u_{(2)}) \otimes u_{(3)})x_0^{h_6-h_1-h_2+n}(x_1 - x_0)^{h_4-h_6-h_3-n} \\ &= \sum_{n \geq 0} o_{n-1}^{\mathcal{Y}^1}(o_{-n-1}^{\mathcal{Y}^2}(u_{(1)} \otimes u_{(2)}) \otimes u_{(3)}) \left(\frac{x_0}{x_1}\right)^{h_6-h_1-h_2+n} \left(1 - \frac{x_0}{x_1}\right)^{h_4-h_6-h_3-n} x_1^{h_4-h_1-h_2-h_3}. \end{aligned}$$

using (5.23). Thus, for each  $n \in \mathbb{N}$ , we can define maps

$$g_n : T(W_1) \otimes T(W_2) \otimes T(W_3) \rightarrow W_4$$

by

$$g_n(u_{(1)} \otimes u_{(2)} \otimes u_{(3)}) = o_{n-1}^{\mathcal{Y}^1}(o_{-n-1}^{\mathcal{Y}^2}(u_{(1)} \otimes u_{(2)}) \otimes u_{(3)})$$

for  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ , and  $u_{(3)} \in T(W_3)$ . These maps are  $\mathfrak{g}$ -homomorphisms by the  $n = 0$  case of (5.4). Now we can take the series  $G(x)$  to be

$$G(x) = \sum_{n \geq 0} g_n^*(u'_{(4)}) x^{h_6 - h_1 - h_2 + n} (1 - x)^{h_4 - h_6 - h_3 - n},$$

where  $*$  denotes the adjoint  $\mathfrak{g}$ -module homomorphism.

To complete the proof, we must show that for any  $\mathfrak{g}$ -homomorphism

$$f : T(W_1) \otimes T(W_2) \otimes T(W_3) \rightarrow T(W_4),$$

we have

$$\langle u'_{(4)}, f(u_{(1)} \otimes u_{(2)} \otimes u_{(3)}) \rangle x^{h_4 - h_1 - h_2 - h_3} = \langle u'_{(4)}, f(x^{\tilde{H}^*}(u_{(1)} \otimes u_{(2)} \otimes u_{(3)})) \rangle.$$

Recall from (2.4) that if  $W$  is an  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -module, then

$$L(0)|_{T(W)} = \frac{1}{2(\ell + h^\vee)} C_{T(W)}$$

where  $C_{T(W)}$  denotes the action on  $T(W)$  of the Casimir operator associated to the invariant bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . Thus since  $L(0)$  is self-adjoint and  $f$  is a  $\mathfrak{g}$ -homomorphism, we have

$$\begin{aligned} & \langle u'_{(4)}, f(u_{(1)} \otimes u_{(2)} \otimes u_{(3)}) \rangle x^{h_4 - h_1 - h_2 - h_3} \\ &= \langle x^{L(0)} u'_{(4)}, f(x^{-L(0)} u_{(1)} \otimes x^{-L(0)} u_{(2)} \otimes x^{-L(0)} u_{(3)}) \rangle \\ &= \langle u'_{(4)}, x^{L(0)} f(x^{-L(0)} u_{(1)} \otimes x^{-L(0)} u_{(2)} \otimes x^{-L(0)} u_{(3)}) \rangle \\ &= \left\langle u'_{(4)}, x^{\frac{C_{T(W_4)}}{2(\ell + h^\vee)}} f \left( x^{-\frac{C_{T(W_1)}}{2(\ell + h^\vee)}} u_{(1)} \otimes x^{-\frac{C_{T(W_2)}}{2(\ell + h^\vee)}} u_{(2)} \otimes x^{-\frac{C_{T(W_3)}}{2(\ell + h^\vee)}} u_{(3)} \right) \right\rangle \\ &= \left\langle u'_{(4)}, f \left( x^{\frac{C_{T(W_1)} \otimes T(W_2) \otimes T(W_3)}} \left( x^{-\frac{C_{T(W_1)}}{2(\ell + h^\vee)}} u_{(1)} \otimes x^{-\frac{C_{T(W_2)}}{2(\ell + h^\vee)}} u_{(2)} \otimes x^{-\frac{C_{T(W_3)}}{2(\ell + h^\vee)}} u_{(3)} \right) \right) \right\rangle. \end{aligned}$$

But it is clear from the action of  $\mathfrak{g}$  on tensor products of  $\mathfrak{g}$ -modules that

$$\begin{aligned} C_{T(W_1) \otimes T(W_2) \otimes T(W_3)} &= C_{T(W_1)} \otimes 1_{T(W_2)} \otimes 1_{T(W_3)} + 1_{T(W_1)} \otimes C_{T(W_2)} \otimes 1_{T(W_3)} \\ &\quad + 1_{T(W_1)} \otimes 1_{T(W_2)} \otimes C_{T(W_3)} + 2(\Omega_{1,2} + \Omega_{1,3} + \Omega_{2,3})^*, \end{aligned}$$

hence by the definition of  $\tilde{H}$ ,

$$x^{\tilde{H}^*}(u_{(1)} \otimes u_{(2)} \otimes u_{(3)}) = x^{\frac{C_{T(W_1)} \otimes T(W_2) \otimes T(W_3)}} \left( x^{-\frac{C_{T(W_1)}}{2(\ell + h^\vee)}} u_{(1)} \otimes x^{-\frac{C_{T(W_2)}}{2(\ell + h^\vee)}} u_{(2)} \otimes x^{-\frac{C_{T(W_3)}}{2(\ell + h^\vee)}} u_{(3)} \right),$$

for any  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ , and  $u_{(3)} \in T(W_3)$ , completing the proof.  $\square$

The significance of the preceding proposition is illustrated by the following:

**Proposition 5.6.** *A series of the form  $x_1^{\tilde{H}} F(\frac{x_2}{x_1})$  where  $F(x) \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^* \{x\}$  solves (5.17)-(5.18) if and only if  $F(x)$  satisfies the formal differential equation*

$$(\ell + h^\vee) \frac{d}{dx} F(x) = \left( \frac{\Omega_{2,3}}{x} - \frac{\Omega_{1,2}}{1-x} \right) F(x). \quad (5.24)$$

*Similarly, a series of the form  $x_1^{\tilde{H}} G(\frac{x_2}{x_1})$  where  $G(x) \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^* \{x\}$  solves (5.21)-(5.22) if and only if  $G(x)$  solves*

$$(\ell + h^\vee) \frac{d}{dx} G(x) = \left( \frac{\Omega_{1,2}}{x} - \frac{\Omega_{2,3}}{1-x} \right) G(x). \quad (5.25)$$

*Proof.* We must first verify that  $\tilde{H}$  commutes with  $\Omega_{1,2}$ ,  $\Omega_{1,3}$ , and  $\Omega_{2,3}$ . It suffices to verify that

$$[\Omega_{1,2}, \Omega_{1,3} + \Omega_{2,3}] = [\Omega_{1,3}, \Omega_{1,2} + \Omega_{2,3}] = [\Omega_{2,3}, \Omega_{1,2} + \Omega_{1,3}] = 0.$$

In fact, for any  $f \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^*$  and  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ ,  $u_{(3)} \in T(W_3)$ , we have

$$\begin{aligned} \langle [\Omega_{1,2}, \Omega_{1,3} + \Omega_{2,3}] f, u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle &= \langle f, [\Omega_{1,3}^* + \Omega_{2,3}^*, \Omega_{1,2}^*](u_{(1)} \otimes u_{(2)} \otimes u_{(3)}) \rangle \\ &= \sum_{i,j=1}^{\dim \mathfrak{g}} \langle f, (\gamma_i \gamma_j u_{(1)} \otimes \gamma_j u_{(2)} + \gamma_j u_{(1)} \otimes \gamma_i \gamma_j u_{(2)}) \otimes \gamma_i u_{(3)} \rangle \\ &\quad - \sum_{i,j=1}^{\dim \mathfrak{g}} \langle f, (\gamma_j \gamma_i u_{(1)} \otimes \gamma_j u_{(2)} + \gamma_j u_{(1)} \otimes \gamma_j \gamma_i u_{(2)}) \otimes \gamma_i u_{(3)} \rangle \\ &= \sum_{i,j=1}^{\dim \mathfrak{g}} \langle f, [\gamma_i, \gamma_j] u_{(1)} \otimes \gamma_j u_{(2)} + \gamma_j u_{(1)} \otimes [\gamma_i, \gamma_j] u_{(2)} \rangle \otimes \gamma_i u_{(3)} \\ &= \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle f, c_{i,j}^k (\gamma_k u_{(1)} \otimes \gamma_j u_{(2)} \otimes \gamma_i u_{(3)}) + c_{i,j}^k (\gamma_j u_{(1)} \otimes \gamma_k u_{(2)} \otimes \gamma_i u_{(3)}) \rangle \\ &= \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle f, (c_{i,j}^k + c_{i,k}^j) (\gamma_k u_{(1)} \otimes \gamma_j u_{(2)} \otimes \gamma_i u_{(3)}) \rangle, \end{aligned}$$

where  $[\gamma_i, \gamma_j] = \sum_{k=1}^{\dim \mathfrak{g}} c_{i,j}^k \gamma_k$  for any  $i, j$ . Now, for any  $i, j$ , and  $k$  we have  $c_{i,j}^k = -c_{j,i}^k$  by skew symmetry and we have  $c_{i,j}^k = c_{j,k}^i$  by the invariance of the form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . This implies that  $c_{i,j}^k = -c_{i,k}^j$  for any  $i, j$ , and  $k$ , which proves that  $[\Omega_{1,2}, \Omega_{1,3} + \Omega_{2,3}] = 0$ . The proofs that  $[\Omega_{1,3}, \Omega_{1,2} + \Omega_{2,3}] = 0$  and  $[\Omega_{2,3}, \Omega_{1,2} + \Omega_{1,3}] = 0$  are similar.

Now suppose  $x_1^{\tilde{H}} F(\frac{x_2}{x_1})$  satisfies (5.17)-(5.18). Using (5.17) and the fact that  $\tilde{H}$  commutes with  $\Omega_{1,2}$  and  $\Omega_{1,3}$ , we have

$$\begin{aligned} (\ell + h^\vee) \frac{\partial}{\partial x_1} \left( x_1^{\tilde{H}} F \left( \frac{x_2}{x_1} \right) \right) &= H x^{\tilde{H}-1} F \left( \frac{x_2}{x_1} \right) - (\ell + h^\vee) x_1^{\tilde{H}-2} x_2 F' \left( \frac{x_2}{x_1} \right) \\ &= x_1^{\tilde{H}-1} \left( \Omega_{1,3} + \frac{\Omega_{1,2}}{1 - x_2/x_1} \right) F \left( \frac{x_2}{x_1} \right), \end{aligned}$$

so that

$$\begin{aligned} (\ell + h^\vee)F' \left( \frac{x_2}{x_1} \right) &= \frac{x_1}{x_2} \left( H - \Omega_{1,3} - \frac{\Omega_{1,2}}{1 - x_2/x_1} \right) F \left( \frac{x_2}{x_1} \right) \\ &= \left( \frac{\Omega_{2,3}}{x_2/x_1} - \frac{\Omega_{1,2}}{1 - x_2/x_1} \right) F \left( \frac{x_2}{x_1} \right). \end{aligned}$$

Thus  $F(x)$  satisfies (5.24). Conversely, if  $F(x)$  satisfies (5.24), reversing the steps above shows that  $x_1^{\tilde{H}} F(\frac{x_2}{x_1})$  satisfies (5.17). We need to show that  $x_1^{\tilde{H}} F(\frac{x_2}{x_1})$  satisfies (5.18). Indeed,

$$\begin{aligned} (\ell + h^\vee) \frac{\partial}{\partial x_2} \left( x_1^{\tilde{H}} F \left( \frac{x_2}{x_1} \right) \right) &= (\ell + h^\vee) x_1^{\tilde{H}-1} F' \left( \frac{x_2}{x_1} \right) \\ &= x_1^{\tilde{H}-1} \left( \frac{\Omega_{2,3}}{x_2/x_1} - \frac{\Omega_{1,2}}{1 - x_2/x_1} \right) F \left( \frac{x_2}{x_1} \right) = \left( \frac{\Omega_{2,3}}{x_2} - \frac{\Omega_{1,2}}{x_1 - x_2} \right) x_1^{\tilde{H}} F \left( \frac{x_2}{x_1} \right), \end{aligned}$$

since  $\tilde{H}$  commutes with  $\Omega_{2,3}$  and  $\Omega_{1,2}$ .

The proof of the second assertion of the proposition is entirely analogous.  $\square$

As an immediate consequence of the preceding two propositions, we have

**Corollary 5.7.** *For any  $u'_{(4)} \in T(W'_4)$ ,  $x_1^{-\tilde{H}} \psi_{y_1, y_2}(u'_{(4)})$  is a series  $F(\frac{x_2}{x_1})$  where  $F(x)$  satisfies (5.24). Similarly,  $x_1^{-\tilde{H}} \tilde{\varphi}_{y^1, y^2}(u'_{(4)})$  is a series  $G(\frac{x_0}{x_1})$  where  $G(x)$  satisfies (5.25).*

### 5.3 Drinfeld associator isomorphisms

In this subsection we study solutions to (5.24) and (5.25) in an abstract setting. We fix a finite-dimensional  $\mathfrak{g}$ -module  $W$  and two diagonalizable  $\mathfrak{g}$ -module endomorphisms  $A$  and  $B$  of  $W$ . Our goal is to obtain a  $\mathfrak{g}$ -module automorphism of  $W$  from solutions to the following one-variable version of the KZ equations:

$$\frac{d\varphi}{dx} = \left( \frac{A}{x} - \frac{B}{1-x} \right) \varphi \quad (5.26)$$

where  $\varphi \in W\{x\}[\log x]$ .

Suppose  $\{\lambda\}$  is the minimal set of eigenvalues of  $A$  such that all eigenvalues of  $A$  are contained in  $\cup_\lambda (\lambda + \mathbb{N})$ . Thus for each  $\lambda$ ,  $\{\lambda + N_j^\lambda\}_{j=0}^{J_\lambda}$  is the set of eigenvalues of  $A$  in  $\lambda + \mathbb{Z}$ , where each  $N_j^\lambda \in \mathbb{N}$  and

$$0 = N_0^\lambda < N_1^\lambda < \dots < N_{J_\lambda}^\lambda.$$

For any eigenvalue  $\mu$  of  $A$ , we use  $\pi_\mu^A$  to denote projection onto the  $\mu$ -eigenspace of  $A$ .

**Proposition 5.8.** *For any  $w \in W$ , there is a unique solution to (5.26) of the form*

$$\varphi_w^A(x) = \sum_\lambda \sum_{j=0}^{J_\lambda} \sum_{i \geq N_j^\lambda} w_{i,j}^{(\lambda)} x^{\lambda+i} (\log x)^j \in W\{x\}[\log x] \quad (5.27)$$

such that for each  $\lambda$ ,  $w_{0,0}^{(\lambda)} = \pi_\lambda^A(w)$  and for each  $j > 0$ ,  $\pi_{\lambda+N_j^\lambda}^A(w_{N_j^\lambda,0}^{(\lambda)}) = \pi_{\lambda+N_j^\lambda}^A(w)$ .

*Proof.* Since no two  $\lambda$ 's are congruent mod  $\mathbb{Z}$ , any series as in (5.27) solves (5.26) if and only if for each  $\lambda$ , the double series over  $j$  and  $i$  in (5.27) solves (5.26). Thus we may fix a  $\lambda$ , and suppose that the  $A$ -eigenvalues in  $\lambda + \mathbb{Z}$  consist of  $\lambda, \lambda + N_1, \dots, \lambda + N_J$ . Then we need to show that any solution to (5.26) of the form

$$\sum_{j=0}^J \sum_{i \geq N_j} w_{i,j} x^{\lambda+i} (\log x)^j \quad (5.28)$$

exists and is uniquely determined by the vectors  $w_{0,0}$  and  $\pi_{\lambda+N_j}^A(w_{N_j,0})$  for  $j > 0$ ; moreover, we need to show that  $w_{0,0}$  is an  $A$ -eigenvector with eigenvalue  $\lambda$ .

We observe that the series (5.28) solves (5.26) if and only if

$$\begin{aligned} & \sum_{j=0}^J \sum_{i \geq N_j} ((\lambda + i - A)w_{i,j} x^{\lambda+i} (\log x)^j + j w_{i,j} x^{\lambda+i} (\log x)^{j-1}) \\ &= -\frac{x}{1-x} \sum_{j=0}^J \sum_{i \geq N_j} B w_{i,j} x^{\lambda+i} (\log x)^j. \end{aligned} \quad (5.29)$$

When we identify powers of  $x$  and  $\log x$  in preceding equation, we see that we must show there is a unique solution to the equations

$$(\lambda + i - A)w_{i,j} + (j+1)w_{i,j+1} = - \sum_{k=N_j}^{i-1} B w_{k,j} \quad (5.30)$$

for  $0 \leq j \leq J$  and  $i \geq N_j$ . It is enough to prove that for any  $j' \leq J$ , the equations (5.30) for  $j \geq j'$  have a solution which is uniquely determined by the vectors  $w_{N_{j'},j'}$  and  $\pi_{\lambda+N_{j'}}^A(w_{N_{j'},j'})$ , and that moreover  $w_{N_{j'},j'}$  is an eigenvector of  $A$  with eigenvalue  $\lambda + N_{j'}$ . We can prove this assertion by downward induction on  $j'$ , starting with the base case  $j' = J$ .

Equations (5.30) for  $j = J$  yield

$$A w_{N_J,J} = (\lambda + N_J) w_{N_J,J}$$

and

$$w_{i,J} = (A - (\lambda + i)1_W)^{-1} \sum_{k=N_J}^{i-1} B w_{k,J}$$

for  $i > N_J$ . This shows that equations (5.30) for  $j = J$  have a solution uniquely determined by  $w_{N_J,J}$ , which must be in the  $(\lambda + N_J)$ -eigenspace for  $A$ . This proves the base case of the induction.

Now suppose that for some  $j' < J$ , equations (5.30) for  $j > j'$  have a solution uniquely determined by the vectors  $w_{N_{j'+1},j'+1}$  and  $\pi_{\lambda+N_{j'+1}}^A(w_{N_{j'+1},j'+1})$  for  $j > j' + 1$ , where  $w_{N_{j'+1},j'+1}$  must be in the  $(\lambda + N_{j'+1})$ -eigenspace of  $A$ . We must show that the coefficients  $w_{i,j'}$  for  $i \geq N_{j'}$  as well as the vectors  $w_{N_{j'+1},j'+1}$  and  $\pi_{\lambda+N_{j'+1}}^A(w_{N_{j'+1},j'+1})$  for  $j > j' + 1$  are uniquely determined by equations (5.30) for  $j = j'$  together with the vectors  $w_{N_{j'},j'}$  and  $\pi_{\lambda+N_{j'}}^A(w_{N_{j'},j'})$  for  $j > j'$ . We must also show that  $w_{N_{j'},j'}$  must be in the  $(\lambda + N_{j'})$ -eigenspace for  $A$ .

In fact, when  $i \neq N_j$  for  $j \geq j'$ , equations (5.30) for  $j = j'$  yield

$$w_{i,j'} = (A - (\lambda + i)1_W)^{-1} \left( (j' + 1)w_{i,j'+1} + \sum_{k=N_{j'}}^{i-1} Bw_{k,j'} \right). \quad (5.31)$$

On the other hand, when  $i = N_{j'}$ , we have

$$Aw_{N_{j'},j'} = (\lambda + N_{j'})w_{N_{j'},j'}, \quad (5.32)$$

and when  $i = N_j$  for  $j > j'$ , projecting (5.30) to the eigenspaces of  $A$  yields

$$\pi_\mu^A(w_{N_j,j'}) = \frac{1}{\mu - \lambda - N_j} \left( (j' + 1)\pi_\mu^A(w_{N_j,j'+1}) + \sum_{k=N_{j'}}^{N_j-1} \pi_\mu^A(Bw_{k,j'}) \right) \quad (5.33)$$

when  $\mu \neq \lambda + N_j$  and

$$\pi_{\lambda+N_j}^A(w_{N_j,j'+1}) = -\frac{1}{j'+1} \sum_{k=N_{j'}}^{N_j-1} \pi_{\lambda+N_j}^A(Bw_{k,j'}). \quad (5.34)$$

Equations (5.31), (5.32), (5.33), and (5.34), together with the induction hypothesis, imply that equations (5.30) for  $j \geq j'$  have a unique solution determined by the vectors  $w_{N_{j'},j'}$  and  $\pi_{\lambda+N_j}^A(w_{N_j,j'})$  for  $j > j'$  whenever  $w_{N_{j'},j'}$  is in the  $(\lambda + N_{j'})$ -eigenspace of  $A$ . This completes the proof.  $\square$

The space of series  $W\{x\}[\log x]$  has an obvious  $\mathfrak{g}$ -module structure, and because  $A$  and  $B$  are  $\mathfrak{g}$ -module endomorphisms, the subspace  $S_{KZ}^{(x)}$  of  $W\{x\}[\log x]$  consisting of solutions to (5.26) is also a  $\mathfrak{g}$ -module. Then the linearity of (5.26) and the fact that  $A$  and  $B$  are  $\mathfrak{g}$ -module endomorphisms implies that the correspondence

$$\varphi_A^{(x)} : w \mapsto \varphi_w^A(x)$$

defines a  $\mathfrak{g}$ -module homomorphism from  $W$  to  $S_{KZ}^{(x)}$ . Our goal, however, is a  $\mathfrak{g}$ -homomorphism from  $W$  to a space of  $W$ -valued analytic functions rather than a space of formal series.

We give  $W$  the weak topology whereby a sequence  $\{w_n\} \subseteq W$  converges to an element  $w \in W$  if and only if for any  $w' \in W^*$ ,  $\langle w', w_n \rangle \rightarrow \langle w', w \rangle$  in the usual topology on  $\mathbb{C}$ . Note that the action of  $\mathfrak{g}$  on  $W$  is continuous because if  $\{w_n\}$  is a sequence converging to  $w \in W$ , then for any  $g \in \mathfrak{g}$  and  $w' \in W^*$ ,

$$\langle w', g \cdot w_n \rangle = -\langle g \cdot w', w_n \rangle \rightarrow -\langle g \cdot w', w \rangle = \langle w', g \cdot w \rangle,$$

so that  $\{g \cdot w_n\}$  converges to  $g \cdot w$ .

The topology on  $W$  allows us to speak of analytic functions from  $\mathbb{C}$  to  $W$ . We use  $S_{KZ}$  to denote the space of analytic  $W$ -valued solutions of

$$\frac{d}{dz}\varphi(z) = \left( \frac{A}{z} - \frac{B}{1-z} \right) \varphi(z) \quad (5.35)$$

on  $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$ . Since  $A$  and  $B$  are  $\mathfrak{g}$ -module homomorphisms,  $S_{KZ}$  is a  $\mathfrak{g}$ -module with the action of  $\mathfrak{g}$  defined by

$$(x \cdot \varphi)(z) = x \cdot \varphi(z)$$

for  $x \in \mathfrak{g}$ ,  $\varphi \in S_{KZ}$ , and  $z \in \mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$ .

Since (5.35) is a linear differential equation with regular singular points at 0, 1, and  $\infty$ , any formal solution

$$\varphi(x) = \sum_{j=0}^J \sum_{n \in \mathbb{C}} w_{j,n} x^n (\log x)^j \in S_{KZ}^{(x)}$$

induces a solution  $\varphi(z) \in S_{KZ}$  which is the analytic extension of the (convergent) series

$$\sum_{j=0}^J \sum_{n \in \mathbb{C}} w_{j,n} z^n (\log z)^j$$

to  $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$  (see for example Appendix B of [42] for an overview of the theory of linear differential equations with regular singular points). Here the complex numbers  $z^n$  and  $\log z$  are determined using some fixed branch of logarithm with a branch cut along the negative real axis. Because the action of  $\mathfrak{g}$  on  $W$  is continuous, the correspondence  $\varphi(x) \mapsto \varphi(z)$  defines a  $\mathfrak{g}$ -module homomorphism from  $S_{KZ}^{(x)}$  to  $S_{KZ}$ . Then we get a  $\mathfrak{g}$ -homomorphism

$$\varphi_A : w \mapsto \varphi_w^A(z)$$

from  $W$  to  $S_{KZ}$ .

**Proposition 5.9.** *The  $\mathfrak{g}$ -homomorphism  $\varphi_A$  is an isomorphism.*

*Proof.* The theory of linear differential equations implies that the dimension of  $S_{KZ}$  equals the dimension of  $W$  so it suffices to show that  $\varphi_A$  is injective. Thus suppose

$$\varphi_w^A(z) = \sum_{\lambda} \sum_{j=0}^{J_{\lambda}} \sum_{i \geq N_j^{\lambda}} w_{i,j}^{(\lambda)} z^{\lambda+i} (\log z)^j = 0$$

for some  $w \in W$ . Then the fact that

$$(\cup_{\lambda} (\lambda + \mathbb{N})) \times \{0, 1, \dots, \max_{\lambda} J_{\lambda}\}$$

is a unique expansion set (see Definition 7.5 and Proposition 7.8 in [30]) implies that  $w_{i,j}^{(\lambda)} = 0$  for all  $\lambda, i, j$ . In particular, for each  $\lambda$ ,  $w_{0,0}^{(\lambda)} = \pi_{\lambda}^A(w) = 0$  and for each  $j > 0$ ,  $\pi_{\lambda+N_j^{\lambda}}^A(w_{N_j^{\lambda},0}^{(\lambda)}) = \pi_{\lambda+N_j^{\lambda}}^A(w) = 0$ . Since  $\pi_{\mu}^A(w) = 0$  for all eigenvalues  $\mu$  of  $A$ ,  $w = 0$ , proving injectivity.  $\square$

Now we consider the operator  $B$  and its eigenvalues. Suppose  $\{\mu\}$  is the minimal set of eigenvalues of  $B$  such that all eigenvalues of  $B$  are contained in  $\cup_{\mu} (\mu + \mathbb{N})$ . Thus for each  $\mu$ ,  $\{\mu + M_k^{\mu}\}_{k=0}^{K_{\mu}^{\mu}}$  is the set of eigenvalues of  $B$  in  $\mu + \mathbb{Z}$ , where each  $M_k^{\mu} \in \mathbb{N}$  and

$$0 = M_0^{\mu} < M_1^{\mu} < \dots < M_{K_{\mu}^{\mu}}^{\mu}.$$

For any eigenvalue  $\lambda$  of  $B$ , we use  $\pi_{\lambda}^B$  to denote projection onto the  $\lambda$ -eigenspace of  $B$ . The proof of Proposition 5.8 yields

**Proposition 5.10.** *For any  $w \in W$ , there is a unique solution to*

$$\frac{d}{dy}\varphi(y) = \left(\frac{B}{y} - \frac{A}{1-y}\right)\varphi(y)$$

*of the form*

$$\varphi_w^B(y) = \sum_{\mu} \sum_{k=0}^{K_{\mu}} \sum_{i \geq M_k^{\mu}} w_{i,k}^{(\mu)} y^{\mu+i} (\log y)^k \in W\{y\}[\log y]$$

*such that for each  $\mu$ ,  $w_{0,0}^{(\mu)} = \pi_{\mu}^B(w)$  and for each  $k > 0$ ,  $\pi_{\mu+M_k^{\lambda}}^B(w_{M_k^{\mu},0}^{(\mu)}) = \pi_{\mu+M_k^{\mu}}^B(w)$ .*

Then the same argument as for the operator  $A$  shows that there is a  $\mathfrak{g}$ -module isomorphism

$$\tilde{\varphi}_B : w \mapsto \varphi_w^B(z)$$

from  $W$  to the space  $\tilde{S}_{KZ}$  of  $W$ -valued analytic solutions of

$$\frac{d}{dz}\varphi(z) = \left(\frac{B}{z} - \frac{A}{1-z}\right)\varphi(z) \quad (5.36)$$

on  $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$ . Observe that replacing  $z$  with  $1-z$  in (5.36) gives (5.35). This means that  $\varphi(z) \in \tilde{S}_{KZ}$  if and only if  $\varphi(1-z) \in S_{KZ}$ , and the correspondence  $\varphi(z) \mapsto \varphi(1-z)$  is a  $\mathfrak{g}$ -module isomorphism from  $\tilde{S}_{KZ}$  to  $S_{KZ}$ .

As a consequence of the preceding considerations, we have a  $\mathfrak{g}$ -module isomorphism

$$\varphi_B : w \mapsto \varphi_w^B(1-z)$$

from  $W$  to  $S_{KZ}$ . This allows us to define the  $\mathfrak{g}$ -module automorphism

$$\Phi_{KZ} = \varphi_B^{-1} \circ \varphi_A$$

of  $W$ , which we call the *Drinfeld associator* for the  $\mathfrak{g}$ -module  $W$  equipped with the diagonalizable  $\mathfrak{g}$ -endomorphisms  $A$  and  $B$ . Observe that  $\Phi_{KZ}$  is defined by the relation

$$\varphi_B \circ \Phi_{KZ} = \varphi_A. \quad (5.37)$$

## 6 The associativity isomorphisms in $\mathbf{D}(\mathfrak{g}, \ell)$

In this section we use the results and constructions of the previous section to describe the associativity isomorphisms in the category  $\mathbf{D}(\mathfrak{g}, \ell)$ .

### 6.1 Associativity of intertwining operators

We recall the convergence and associativity of intertwining operators in  $L_{\hat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$ , which was proved in Theorem 3.8 in [25] using the KZ equations:

**Theorem 6.1.** *Suppose  $W_1, W_2, W_3$ , and  $W_4$  are  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules.*

1. For any  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module  $M_1$  and intertwining operators  $\mathcal{Y}_1$  of type  $\binom{W_4}{W_1 M_1}$  and  $\mathcal{Y}_2$  of type  $\binom{M_1}{W_2 W_3}$ , the double series

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \rangle$$

converges absolutely for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w'_{(4)} \in W'_4$  and  $|z_1| > |z_2| > 0$ .

2. For any  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module  $M_2$  and intertwining operators  $\mathcal{Y}^1$  of type  $\binom{W_4}{M_2 W_3}$  and  $\mathcal{Y}^2$  of type  $\binom{M_2}{W_1 W_2}$ , the double series

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_0) w_{(2)}, z_2) w_{(3)} \rangle$$

converges absolutely for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w'_{(4)} \in W'_4$  and  $|z_2| > |z_0| > 0$ .

3. For any  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module  $M_1$  and intertwining operators  $\mathcal{Y}_1$  of type  $\binom{W_4}{W_1 M_1}$  and  $\mathcal{Y}_2$  of type  $\binom{M_1}{W_2 W_3}$ , there is an  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module  $M_2$  and intertwining operators  $\mathcal{Y}^1$  of type  $\binom{W_4}{M_2 W_3}$  and  $\mathcal{Y}^2$  of type  $\binom{M_2}{W_1 W_2}$  such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \rangle = \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2) w_{(2)}, z_2) w_{(3)} \rangle$$

for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w'_{(4)} \in W'_4$  and  $|z_1| > |z_2| > |z_1 - z_2| > 0$ .

4. For any  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module  $M_2$  and intertwining operators  $\mathcal{Y}^1$  of type  $\binom{W_4}{M_2 W_3}$  and  $\mathcal{Y}^2$  of type  $\binom{M_2}{W_1 W_2}$ , there is an  $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ -module  $M_1$  and intertwining operators  $\mathcal{Y}_1$  of type  $\binom{W_4}{W_1 M_1}$  and  $\mathcal{Y}_2$  of type  $\binom{M_1}{W_2 W_3}$  such that

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_0) w_{(2)}, z_2) w_{(3)} \rangle = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_0 + z_2) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \rangle$$

for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w'_{(4)} \in W'_4$  and  $|z_0 + z_2| > |z_2| > |z_0| > 0$ .

**Remark 6.2.** It is easy to see from the universal property of  $P(z_2)$ -tensor products that the module  $M_1$  in part 4 of Theorem 6.1 can be taken to be  $W_2 \boxtimes_{P(z_2)} W_3$  and that  $\mathcal{Y}_2$  can be taken to be the corresponding tensor product intertwining operator. Then the intertwining operator  $\mathcal{Y}_1$  is uniquely determined by these choices (see Corollary 9.30 in [31]). An analogous remark holds for the module  $M_2$  and intertwining operators  $\mathcal{Y}^1$  and  $\mathcal{Y}^2$  in part 3 of the theorem.

We now suppose  $\mathcal{Y}_1 \in \mathcal{V}_{W_1 M_1}^{W_4}$ ,  $\mathcal{Y}_2 \in \mathcal{V}_{W_2 W_3}^{M_1}$ ,  $\mathcal{Y}^1 \in \mathcal{V}_{M_2 W_3}^{W_4}$ , and  $\mathcal{Y}^2 \in \mathcal{V}_{W_1 W_2}^{M_2}$  satisfy the associativity relations of Theorem 6.1. For  $u'_{(4)} \in T(W'_4)$ , we recall from Subsection 5.1 the series

$$\psi_{\mathcal{Y}_1, \mathcal{Y}_2}(u'_{(4)}) \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^* \{x_1, x_2\}$$

and

$$\widetilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^* \{x_0, x_1\}.$$

By Corollary 5.7, we know

$$\psi_{\mathcal{Y}_1, \mathcal{Y}_2}(u'_{(4)}) = x_1^{\tilde{H}} F\left(\frac{x_2}{x_1}\right) \quad \text{and} \quad \tilde{\varphi}_{\mathcal{Y}^1, \mathcal{Y}^2}(u'_{(4)}) = x_1^{\tilde{H}} G\left(\frac{x_0}{x_1}\right)$$

where  $F(x)$  satisfies (5.24) and  $G(x)$  satisfies (5.25).

We now take the  $\mathfrak{g}$ -module  $W$  of Subsection 5.3 to be  $(T(W_1) \otimes T(W_2) \otimes T(W_3))^*$  and we take the  $\mathfrak{g}$ -module endomorphisms in (5.26) to be  $A = \tilde{\Omega}_{1,2} = (\ell + h^\vee)^{-1} \Omega_{1,2}$  and  $B = \tilde{\Omega}_{2,3} = (\ell + h^\vee)^{-1} \Omega_{2,3}$ . It is easy to see that  $\tilde{\Omega}_{1,2}$  and  $\tilde{\Omega}_{2,3}$  are diagonalizable. For example,

$$\tilde{\Omega}_{1,2}^* = \frac{1}{2(\ell + h^\vee)} (C_{T(W_1) \otimes T(W_2)} - C_{T(W_1)} \otimes 1_{T(W_2)} - 1_{T(W_1)} \otimes C_{T(W_2)}) \otimes 1_{T(W_3)}, \quad (6.1)$$

where as earlier  $C_U$  for a  $\mathfrak{g}$ -module  $U$  is the Casimir operator on  $U$ . Since  $C_{T(W_1) \otimes T(W_2)}$ ,  $C_{T(W_1)} \otimes 1_{T(W_2)}$ , and  $1_{T(W_1)} \otimes C_{T(W_2)}$  are all diagonalizable and commute, they are simultaneously diagonalizable. Hence  $\tilde{\Omega}_{1,2}^*$  and  $\tilde{\Omega}_{1,2}$  are diagonalizable. Similarly,  $\tilde{\Omega}_{2,3}$  is diagonalizable. Thus we have the  $\mathfrak{g}$ -module automorphism  $\Phi_{KZ} = \varphi_{\tilde{\Omega}_{2,3}}^{-1} \circ \varphi_{\tilde{\Omega}_{1,2}}$  of  $(T(W_1) \otimes T(W_2) \otimes T(W_3))^*$ .

For any  $u'_{(4)} \in T(W'_4)$ , we define linear functionals

$$F_{Pr}(u'_{(4)}), F_{It}(u'_{(4)}) \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^*$$

by

$$\langle F_{Pr}(u'_{(4)}), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle = \langle u'_{(4)}, o_{-1}^{\mathcal{Y}_1}(u_{(1)} \otimes o_{-1}^{\mathcal{Y}_2}(u_{(2)} \otimes u_{(3)})) \rangle$$

and

$$\langle F_{It}(u'_{(4)}), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle = \langle u'_{(4)}, o_{-1}^{\mathcal{Y}_1}(o_{-1}^{\mathcal{Y}_2}(u_{(1)} \otimes u_{(2)}) \otimes u_{(3)}) \rangle$$

for  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ , and  $u_{(3)} \in T(W_3)$ . The main result of this subsection is:

**Theorem 6.3.** *For any  $u'_{(4)} \in T(W'_4)$ ,  $\Phi_{KZ}(F_{It}(u'_{(4)})) = F_{Pr}(u'_{(4)})$ .*

*Proof.* The first step in the proof is to show that for any  $u'_{(4)} \in T(W'_4)$ ,

$$F(x) = \varphi_{F_{Pr}(u'_{(4)})}^{\tilde{\Omega}_{2,3}}(x), \quad (6.2)$$

using the notation of Proposition 5.10, and

$$G(x) = \varphi_{F_{It}(u'_{(4)})}^{\tilde{\Omega}_{1,2}}(x), \quad (6.3)$$

using the notation of Proposition 5.8. Since  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules are completely reducible, we may without loss of generality assume that the  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules  $W_1, W_2, W_3, W_4, M_1$ , and  $M_2$  are irreducible with lowest conformal weights  $h_1, \dots, h_6$ , respectively. We prove (6.3), the proof of (6.2) being similar and indeed slightly simpler.

Since  $G(x)$  satisfies (5.25), certainly  $G(x) = \varphi_{w_{It}}^{\tilde{\Omega}_{1,2}}(x)$  for some  $w_{It} \in (T(W_1) \otimes T(W_2) \otimes T(W_3))^*$ . We recall from the proof of Proposition 5.5 that

$$\begin{aligned} G(x) &= \sum_{n \geq 0} g_n^*(u'_{(4)}) x^{h_6 - h_1 - h_2} (1 - x)^{h_4 - h_6 - h_3 - n} \\ &= \sum_{n \geq 0} \sum_{m \geq 0} (-1)^m \binom{h_4 - h_6 - h_3 - n}{m} g_n^*(u'_{(4)}) x^{h_6 - h_1 - h_2 + m + n} \\ &= \sum_{n \geq 0} \left( \sum_{m=0}^n (-1)^m \binom{h_4 - h_6 - h_3 - n + m}{m} g_{n-m}^*(u'_{(4)}) \right) x^{h_6 - h_1 - h_2 + n} \end{aligned} \quad (6.4)$$

where  $g_n^*$  is the adjoint of

$$g_n = o_{-n-1}^{\mathcal{Y}^1}(o_{n-1}^{\mathcal{Y}^2}(\cdot \otimes \cdot) \otimes \cdot).$$

Suppose we use  $\{h_6 - h_1 - h_2 + N_j\}_{j=0}^J$ , where  $N_0 = 0$  and each  $N_j \in \mathbb{N}$ , to denote the eigenvalues of  $\tilde{\Omega}_{1,2}$  in  $h_6 - h_1 - h_2 + \mathbb{N}$ . Then we see from Proposition 5.8 and (6.4) that

$$w_{It} = \sum_{j=0}^J \sum_{m=0}^{N_j} (-1)^m \binom{h_4 - h_6 - h_3 - N_j + m}{m} \pi_{h_6 - h_1 - h_2 + N_j}^{\tilde{\Omega}_{1,2}}(g_{N_j - m}^*(u'_{(4)})).$$

Now,  $F_{It}(u'_{(4)}) = g_0^*(u'_{(4)})$  is an eigenvector of  $\tilde{\Omega}_{1,2}$  with eigenvalue  $h_6 - h_1 - h_2$ . Indeed, for  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$  and  $u_{(3)} \in T(W_3)$ , using (6.1), (2.4), and the fact that Casimir operators commute with the  $\mathfrak{g}$ -module homomorphism  $o_{-1}^{\mathcal{Y}^2}$ , we have

$$\begin{aligned} &\langle \tilde{\Omega}_{1,2}(F_{It}(u'_{(4)})), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle \\ &= \frac{1}{2(\ell + h^\vee)} \langle u'_{(4)}, o_{-1}^{\mathcal{Y}^1}(o_{-1}^{\mathcal{Y}^2}(C_{T(W_1) \otimes T(W_2)}(u_{(1)} \otimes u_{(2)})) \otimes u_{(3)}) \rangle \\ &\quad - \frac{1}{2(\ell + h^\vee)} \langle u'_{(4)}, o_{-1}^{\mathcal{Y}^1}(o_{-1}^{\mathcal{Y}^2}(C_{T(W_1)}(u_{(1)}) \otimes u_{(2)} + u_{(1)} \otimes C_{T(W_2)}(u_{(2)})) \otimes u_{(3)}) \rangle \\ &= \frac{1}{2(\ell + h^\vee)} \langle u'_{(4)}, o_{-1}^{\mathcal{Y}^1}(C_{T(M_2)} o_{-1}^{\mathcal{Y}^2}(u_{(1)} \otimes u_{(2)}) \otimes u_{(3)}) \rangle \\ &\quad - (h_1 + h_2) \langle F_{It}(u'_{(4)}), u_1 \otimes u_2 \otimes u_3 \rangle \\ &= (h_6 - h_1 - h_2) \langle F_{It}(u'_{(4)}), u_1 \otimes u_2 \otimes u_3 \rangle. \end{aligned}$$

Thus we have

$$w_{It} - F_{It}(u'_{(4)}) = \sum_{j=1}^J \sum_{m=0}^{N_j} (-1)^m \binom{h_4 - h_6 - h_3 - N_j + m}{m} \pi_{h_6 - h_1 - h_2 + N_j}^{\tilde{\Omega}_{1,2}}(g_{N_j - m}^*(u'_{(4)})).$$

Consequently, it is enough to show that  $\pi_{h_6 - h_1 - h_2 + N_j}^{\tilde{\Omega}_{1,2}}(g_{N_j - m}^*(u'_{(4)})) = 0$  for  $j \geq 1$  and  $0 \leq m \leq N_j$ .

Let us use  $\pi_h^{C_{T(W_1) \otimes T(W_2)}}$  for an eigenvalue  $h$  of  $C_{T(W_1) \otimes T(W_2)}$  to denote projection from  $T(W_1) \otimes T(W_2)$  to the  $h$ -eigenspace of  $C_{T(W_1) \otimes T(W_2)}$ . Then

$$\begin{aligned} & \langle \pi_{h_6-h_1-h_2+N_j}^{\tilde{\Omega}_{1,2}}(g_{N_j-m}^*(u'_{(4)})), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle \\ &= \langle u'_{(4)}, o_{-N_j+m-1}^{\mathcal{Y}^1}(o_{N_j-m-1}^{\mathcal{Y}^2}(\pi_{2(\ell+h^\vee)(h_6+N_j)}^{C_{T(W_1) \otimes T(W_2)}}(u_{(1)} \otimes u_{(2)})) \otimes u_{(3)} \rangle \end{aligned}$$

Note that the image of  $\pi_{2(\ell+h^\vee)(h_6+N_j)}^{C_{T(W_1) \otimes T(W_2)}}$  is the sum of all  $\mathfrak{g}$ -submodules of  $T(W_1) \otimes T(W_2)$  which are isomorphic to some  $L_{\lambda'}$  where  $\lambda'$  is a dominant integral weight of  $\mathfrak{g}$  that satisfies  $h_{\lambda',\ell} = h_6 + N_j$ . Note also that the  $n = 0$  case of (5.4) implies that  $o_{N_j-m-1}^{\mathcal{Y}^2}$  is a  $\mathfrak{g}$ -homomorphism, which maps  $T(W_1) \otimes T(W_2)$  into  $(M_2)_{(h_6+N_j-m)}$ . Thus to show that  $\pi_{h_6-h_1-h_2+N_j}^{\tilde{\Omega}_{1,2}}(g_{N_j-m}^*(u'_{(4)})) = 0$  for  $j \geq 1$  and  $0 \leq m \leq N_j$ , it is enough to show that  $(M_2)_{(h_6+N_j-m)}$  cannot contain a  $\mathfrak{g}$ -submodule with highest weight  $\lambda'$  such that  $h_{\lambda',\ell} = h_6 + N_j$ .

Suppose that  $T(M_2)$  is an irreducible  $\mathfrak{g}$ -module with highest weight  $\lambda$ , so that  $\langle \lambda, \theta \rangle \leq \ell$ . Then  $h_6 = h_{\lambda,\ell}$ ; also, recalling from Subsection 2.2 that a  $\widehat{\mathfrak{g}}$ -module becomes a  $\widetilde{\mathfrak{g}}$ -module on which the derivation  $\mathbf{d}$  acts as  $-L(0)$ , we see that  $M_2$  as a  $\widetilde{\mathfrak{g}}$ -module is isomorphic to  $L(\Lambda)$  where

$$\Lambda = \lambda + \ell \mathbf{k}' - h_{\lambda,\ell} \mathbf{d}'.$$

We need to show that for a dominant integral weight  $\lambda'$  of  $\mathfrak{g}$  such that  $h_{\lambda',\ell} = h_{\lambda,\ell} + N_j$  for  $j \geq 1$ , and for  $0 \leq m \leq N_j$ ,  $L(\Lambda)$  cannot have

$$\Lambda' = \lambda' + \ell \mathbf{k}' - (h_{\lambda,\ell} + N_j - m) \mathbf{d}' = \lambda' + \ell \mathbf{k}' - (h_{\lambda',\ell} - m) \mathbf{d}'$$

as a weight. But this follows immediately from Theorem 2.6, completing the proof that  $w_{It} = F_{It}(u'_{(4)})$ .

Now that we know  $F(x) = \varphi_{F_{Pr}(u'_{(4)})}^{\tilde{\Omega}_{2,3}}(x)$  and  $G(x) = \varphi_{F_{It}(u'_{(4)})}^{\tilde{\Omega}_{1,2}}(x)$ , we can prove that  $\Phi_{KZ}(F_{It}(u'_{(4)})) = F_{Pr}(u'_{(4)})$  for  $u'_{(4)} \in T(W_4)^*$ . For this, it is enough to show that

$$\varphi_{\Phi_{KZ}(F_{It}(u'_{(4)}))}^{\tilde{\Omega}_{2,3}}(z) = \varphi_{F_{Pr}(u'_{(4)})}^{\tilde{\Omega}_{2,3}}(z)$$

for all  $z$  contained in some non-empty open set  $U$  of  $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$ . In fact, we take  $U$  to be the set of  $z \in \mathbb{C}$  such that  $|z| < 1$  and  $\operatorname{Re} z > \frac{1}{2}$ . Equivalently,  $U$  is the set of  $z \in \mathbb{C}$  which satisfy

$$1 > |z| > |1 - z| > 0.$$

We recall from Subsection 5.3 the isomorphisms

$$\varphi_{\tilde{\Omega}_{1,2}} : w \mapsto \varphi_w^{\tilde{\Omega}_{1,2}}(z)$$

and

$$\varphi_{\tilde{\Omega}_{2,3}} : w \mapsto \varphi_w^{\tilde{\Omega}_{2,3}}(1 - z)$$

from  $(T(W_1) \otimes T(W_2) \otimes T(W_3))^*$  to  $S_{KZ}$ .

Now, using (5.37), Corollary 5.7, (6.1), (6.2), and (6.3), we see that for any  $z \in U$ ,

$$\begin{aligned}
& \langle 1^{\tilde{H}} \varphi_{\Phi_{KZ}(F_{It}(u'_{(4)}))}^{\tilde{\Omega}_{2,3}}(z), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle \\
&= \langle 1^{\tilde{H}}((\varphi_{\tilde{\Omega}_{2,3}} \circ \Phi_{KZ})(F_{It}(u'_{(4)})))(1-z), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle \\
&= \langle 1^{\tilde{H}}(\varphi_{\tilde{\Omega}_{1,2}}(F_{It}(u'_{(4)})))(1-z), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle \\
&= \langle u'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(u_{(1)}, 1-z)u_{(2)}, z)u_{(3)} \rangle \\
&= \langle u'_{(4)}, \mathcal{Y}_1(u_{(1)}, 1)\mathcal{Y}_2(u_{(2)}, z)u_{(3)} \rangle \\
&= \langle 1^{\tilde{H}} \varphi_{F_{Pr}(u'_{(4)})}^{\tilde{\Omega}_{2,3}}(z), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle
\end{aligned}$$

for any  $u_{(1)} \in T(W_1)$ ,  $u_{(2)} \in T(W_2)$ , and  $u_{(3)} \in T(W_3)$ . Since  $1^{\tilde{H}}$  is invertible, this completes the proof of the theorem.  $\square$

## 6.2 The associativity isomorphisms in $L_{\hat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$ and their restrictions to top levels

We now recall from [17] (see also [31]) the construction of the associativity isomorphisms in the tensor category  $L_{\hat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$ , using associativity of intertwining operators. We recall from Subsection 4.4 that for  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules  $W_1$  and  $W_2$  and  $z \in \mathbb{C}^\times$ , we have chosen

$$(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)}) = (W_1 \boxtimes_{P(1)} W_2 = S(A(W_1) \otimes_{A(L_{\hat{\mathfrak{g}}}(\ell, 0))} T(W_2)), \mathcal{Y}_{\boxtimes}^{1,2}(\cdot, e^{\log z} \cdot)),$$

where  $\mathcal{Y}_{\boxtimes}^{1,2}$  is the intertwining operator of type  $\left( \begin{smallmatrix} S(A(W_1) \otimes_{A(L_{\hat{\mathfrak{g}}}(\ell, 0))} T(W_2)) \\ W_1 \quad W_2 \end{smallmatrix} \right)$  induced from the identity map on  $A(W_1) \otimes_{A(L_{\hat{\mathfrak{g}}}(\ell, 0))} T(W_2)$ . In particular, we have chosen the tensor products  $W_1 \boxtimes_{P(z)} W_2$  to be the same as modules for all  $z \in \mathbb{C}^\times$ . We also recall that we have chosen  $\boxtimes_{P(1)}$  as the tensor product bifunctor for  $L_{\hat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$  as a tensor category.

Now suppose  $W_1$ ,  $W_2$ , and  $W_3$  are three  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules. We have the tensor product intertwining operators  $\mathcal{Y}_{\boxtimes}^{2,3}$  of type  $\left( \begin{smallmatrix} W_1 \boxtimes_{P(1)} W_2 \\ W_1 \quad W_2 \end{smallmatrix} \right)$ ,  $\mathcal{Y}_{\boxtimes}^{1,2 \otimes 3}$  of type  $\left( \begin{smallmatrix} W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(1)} W_3) \\ W_1 \quad W_2 \boxtimes_{P(1)} W_3 \end{smallmatrix} \right)$ ,  $\mathcal{Y}_{\boxtimes}^{1,2}$  of type  $\left( \begin{smallmatrix} W_1 \boxtimes_{P(1)} W_2 \\ W_1 \quad W_2 \end{smallmatrix} \right)$ , and  $\mathcal{Y}_{\boxtimes}^{1 \otimes 2, 3}$  of type  $\left( \begin{smallmatrix} (W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(1)} W_3 \\ W_1 \boxtimes_{P(1)} W_2 \quad W_3 \end{smallmatrix} \right)$ . Then part 4 of Theorem 6.1 and Remark 6.2 imply that there is a unique intertwining operator  $\mathcal{Y}$  of type  $\left( \begin{smallmatrix} (W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(1)} W_3 \\ W_1 \quad W_2 \boxtimes_{P(1)} W_3 \end{smallmatrix} \right)$  such that

$$\langle w'_{(4)}, \mathcal{Y}_{\boxtimes}^{1 \otimes 2, 3}(\mathcal{Y}_{\boxtimes}^{1,2}(w_{(1)}, z_1 - z_2)w_{(2)}, z_2)w_{(3)} \rangle = \langle w'_{(4)}, \mathcal{Y}(w_{(1)}, z_1)\mathcal{Y}_{\boxtimes}^{2,3}(w_{(2)}, z_2)w_{(3)} \rangle \quad (6.5)$$

for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w'_{(4)} \in W_4$ , and  $z_1, z_2 \in \mathbb{C}^\times$  satisfying

$$|z_1| > |z_2| > |z_1 - z_2| > 0; \quad (6.6)$$

as usual, we substitute formal variables with complex numbers using the branch  $\log z$  of the logarithm.

Now the universal property of the  $P(z_1)$ -tensor product implies that there is a unique  $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -homomorphism

$$\mathcal{A}_{z_1, z_2} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(1)} W_3) = W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(1)} W_3) \rightarrow (W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(1)} W_3$$

such that

$$\mathcal{A}_{z_1, z_2} \circ \mathcal{Y}_{\boxtimes}^{1, 2 \otimes 3}(\cdot, z_1) \cdot = \mathcal{Y}(\cdot, z_1) \cdot. \quad (6.7)$$

In particular, (6.5) implies

$$\langle w'_{(4)}, (w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle = \langle w'_{(4)}, \mathcal{A}_{z_1, z_2}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \rangle \quad (6.8)$$

for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ , and  $w'_{(4)} \in W_4$ . It is easy to see that projections to the conformal weight spaces of elements  $\mathcal{Y}_{\boxtimes}^{1, 2 \otimes 3}(w_{(1)}, z_1)w_{(2,3)}$  for  $w_{(1)} \in W_1$ ,  $w_{(2,3)} \in W_2 \boxtimes_{P(1)} W_3$  span  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(1)} W_3) = W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(1)} W_3)$  (see for instance Lemma 4.9 of [17] or Proposition 4.23 in [28]). This means that  $W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(1)} W_3)$  is also spanned by coefficients of powers of  $x$  in series  $\mathcal{Y}_{\boxtimes}^{1, 2 \otimes 3}(w_{(1)}, x)w_{(2,3)}$  for  $w_{(1)} \in W_1$ ,  $w_{(2,3)} \in W_2 \boxtimes_{P(1)} W_3$ . Thus the homomorphism  $\mathcal{A}_{z_1, z_2}$  is completely determined by the condition

$$\mathcal{A}_{z_1, z_2} \circ \mathcal{Y}_{\boxtimes}^{1, 2 \otimes 3}(\cdot, x) \cdot = \mathcal{Y}(\cdot, x) \cdot,$$

which follows from Proposition 3.6 and (6.7). Consequently, since  $\mathcal{Y}_{\boxtimes}^{1, 2 \otimes 3}$  and  $\mathcal{Y}$  are independent of  $z_1$  and  $z_2$ , so is  $\mathcal{A}_{z_1, z_2}$ . In fact,  $\mathcal{A}_{z_1, z_2}$  is the associativity isomorphism  $\mathcal{A}_{W_1, W_2, W_3}$ .

**Remark 6.4.** The invertibility of  $\mathcal{A}$  follows from part 3 Theorem 6.1, which implies that there is a homomorphism

$$\mathcal{A}_{W_1, W_2, W_3}^{-1} : (W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(1)} W_3 \rightarrow W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(1)} W_3)$$

which satisfies

$$\langle w'_{(4)}, \mathcal{A}_{W_1, W_2, W_3}^{-1}((w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}) \rangle = \langle w'_{(4)}, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle$$

for any  $z_1, z_2 \in \mathbb{C}^\times$  which satisfy (6.6). This condition together with (6.8) shows that  $\mathcal{A}_{W_1, W_2, W_3}^{-1}$  is in fact the inverse of  $\mathcal{A}_{W_1, W_2, W_3}$ .

Now suppose  $U_1$ ,  $U_2$ , and  $U_3$  are objects of  $\mathbf{D}(\mathfrak{g}, \ell)$ . We will use Theorem 6.3 and (6.8) to describe the associativity isomorphism  $\mathcal{A}_{U_1, U_2, U_3}$ . First we introduce some notation: we use  $W_{U_1, (U_2, U_3)}^{(\ell)}$  to denote the kernel of the composition of the natural projections

$$U_1 \otimes U_2 \otimes U_3 \rightarrow U_1 \otimes (U_2 \boxtimes U_3) \rightarrow U_1 \boxtimes (U_2 \boxtimes U_3),$$

and similarly we use  $W_{(U_1, U_2), U_3}^{(\ell)}$  to denote the kernel of the composition of the natural projections

$$U_1 \otimes U_2 \otimes U_3 \rightarrow (U_1 \boxtimes U_2) \otimes U_3 \rightarrow (U_1 \boxtimes U_2) \boxtimes U_3.$$

Thus using the natural isomorphism of Proposition 4.8, we have natural isomorphisms

$$\Psi_{U_1, (U_2, U_3)} : (U_1 \otimes U_2 \otimes U_3) / W_{U_1, (U_2, U_3)}^{(\ell)} \rightarrow T(S(U_1) \boxtimes_{P(1)} (S(U_2) \boxtimes_{P(1)} S(U_3)))$$

and

$$\Psi_{(U_1, U_2), U_3} : (U_1 \otimes U_2 \otimes U_3) / W_{(U_1, U_2), U_3}^{(\ell)} \rightarrow T((S(U_1) \boxtimes_{P(1)} S(U_2)) \boxtimes_{P(1)} S(U_3)).$$

Then from the discussion in Subsection 4.3, we can identify  $\mathcal{A}_{U_1, U_2, U_3}$  with the isomorphism

$$\begin{aligned} \Psi_{(U_1, U_2), U_3}^{-1} \circ T(\mathcal{A}_{S(U_1), S(U_2), S(U_3)}) \circ \Psi_{U_1, (U_2, U_3)} : (U_1 \otimes U_2 \otimes U_3) / W_{U_1, (U_2, U_3)}^{(\ell)} \\ \rightarrow (U_1 \otimes U_2 \otimes U_3) / W_{(U_1, U_2), U_3}^{(\ell)} \end{aligned}$$

Using the notation of Subsection 6.1 and taking  $W_i = S(U_i)$  for  $i = 1, 2, 3$ ,  $\mathcal{Y}^1 = \mathcal{Y}_{\boxtimes}^{1 \otimes 2, 3}$ ,  $\mathcal{Y}^2 = \mathcal{Y}_{\boxtimes}^{1, 2}$ ,  $\mathcal{Y}_1 = \mathcal{A}_{S(U_1), S(U_2), S(U_3)} \circ \mathcal{Y}_{\boxtimes}^{1, 2 \otimes 3}$ , and  $\mathcal{Y}_2 = \mathcal{Y}_{\boxtimes}^{2, 3}$ , we see from the definitions (in particular (4.11)) that

$$\Psi_{(U_1, U_2), U_3}(u_{(1)} \otimes u_{(2)} \otimes u_{(3)} + W_{(U_1, U_2), U_3}^{(\ell)}) = o_{-1}^{\mathcal{Y}_{\boxtimes}^{1 \otimes 2, 3}}(o_{-1}^{\mathcal{Y}_{\boxtimes}^{1, 2}}(u_{(1)} \otimes u_{(2)}) \otimes u_{(3)})$$

and

$$\Psi_{U_1, (U_2, U_3)}(u_{(1)} \otimes u_{(2)} \otimes u_{(3)} + W_{U_1, (U_2, U_3)}^{(\ell)}) = o_{-1}^{\mathcal{Y}_{\boxtimes}^{1, 2 \otimes 3}}(u_{(1)} \otimes o_{-1}^{\mathcal{Y}_{\boxtimes}^{2, 3}}(u_{(2)} \otimes u_{(3)})),$$

for any  $u_{(1)} \in U_1$ ,  $u_{(2)} \in U_2$ ,  $u_{(3)} \in U_3$ , so that

$$\langle F_{It}(u'_{(4)}), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle = \langle u'_{(4)}, \Psi_{(U_1, U_2), U_3}(u_{(1)} \otimes u_{(2)} \otimes u_{(3)} + W_{(U_1, U_2), U_3}^{(\ell)}) \rangle$$

and

$$\begin{aligned} \langle F_{Pr}(u'_{(4)}), u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \rangle \\ = \langle u'_{(4)}, (T(\mathcal{A}_{S(U_1), S(U_2), S(U_3)}) \circ \Psi_{U_1, (U_2, U_3)})(u_{(1)} \otimes u_{(2)} \otimes u_{(3)} + W_{U_1, (U_2, U_3)}^{(\ell)}) \rangle \end{aligned}$$

for any  $u'_{(4)} \in T((W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(1)} W_3)^*$ . Then we immediately obtain from Theorem 6.3 the following description of the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$ :

**Theorem 6.5.** *Under the identifications of  $U_1 \boxtimes (U_2 \boxtimes U_3)$  with  $(U_1 \otimes U_2 \otimes U_3) / W_{U_1, (U_2, U_3)}^{(\ell)}$  and  $(U_1 \boxtimes U_2) \boxtimes U_3$  with  $(U_1 \otimes U_2 \otimes U_3) / W_{(U_1, U_2), U_3}^{(\ell)}$ ,*

$$\mathcal{A}_{U_1, U_2, U_3}(u_{(1)} \otimes u_{(2)} \otimes u_{(3)} + W_{U_1, (U_2, U_3)}^{(\ell)}) = \Phi_{KZ}^*(u_{(1)} \otimes u_{(2)} \otimes u_{(3)}) + W_{(U_1, U_2), U_3}^{(\ell)},$$

for any  $u_{(1)} \in U_1$ ,  $u_{(2)} \in U_2$ , and  $u_{(3)} \in U_3$ , where  $\Phi_{KZ}^*$  is the automorphism of  $U_1 \otimes U_2 \otimes U_3$  adjoint to  $\Phi_{KZ}$ .

**Remark 6.6.** Note that one non-trivial implication of Theorem 6.5 is that

$$\Phi_{KZ}^*(W_{U_1, (U_2, U_3)}^{(\ell)}) = W_{(U_1, U_2), U_3}^{(\ell)}.$$

It seems to be difficult to prove this directly, without using the associativity isomorphisms in  $L_{\widehat{\mathfrak{g}}}(\ell, 0) - \mathbf{mod}$ . Also, the fact that  $W_{U_1, (U_2, U_3)}^{(\ell)}$  and  $W_{(U_1, U_2), U_3}^{(\ell)}$  are generally distinct subspaces of  $U_1 \otimes U_2 \otimes U_3$  shows why the associativity isomorphisms in  $\mathbf{D}(\mathfrak{g}, \ell)$  must be induced by ordinarily non-trivial isomorphisms such as  $\Phi_{KZ}^*$ .

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